Analogs of Cuntz algebras on L^p spaces

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24 April 2013

Graph algebras: Bridges between graph C*-algebras and Leavitt path algebras)

Banff International Research Station

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Notation for Leavitt algebras

From now on, we always take the base field to be $\mathbb{C}$.

Definition

Let $d \in \{2, 3, 4, \ldots\}$. We define the *Leavitt algebra* L_d to be the universal complex associative algebra on generators $s_1, s_2, \ldots, s_d, t_1, t_2, \ldots, t_d$ satisfying the relations:

$$t_j s_j = 1 \qquad \text{for } j \in \{1, 2, \ldots, d\},$$

satisfying the relations:

$$t_j s_k = 0 \qquad \text{for } j, k \in \{1, 2, \ldots, d\} \text{ with } j \neq k,$$

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From now on (with very occasional exceptions), all representations of L_d will be taken to be unital.

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1. As before: Multiplication by functions of absolute value 1.
2. As before: Let $Y_0 \subset Y$. Let $h\colon X \to Y_0$ be a measure preserving bijection. Then define $v\colon L^p(X, \mu) \to L^p(Y, \nu)$ by
$$(v\xi)(y) = \begin{cases} \xi(h^{-1}(y)) & y \in Y_0 \\ 0 & y \notin Y_0. \end{cases}$$
3. (A modification of the preceding example.) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be σ-finite measure spaces. Let $h\colon X \to Y_0$ be a bimeasurable bijection such that for $E \subset X$ we have $\nu(h(E)) = 0$ if and only if $\mu(E) = 0$. Then define $v\colon L^p(X, \mu) \to L^p(Y, \nu)$ by
$$(v\xi)(y) = \begin{cases} \left(\left[\frac{dh_*(\mu)}{d\nu} \right](y) \right)^{1/p} \xi(h^{-1}(y)) & y \in Y_0 \\ 0 & y \notin Y_0. \end{cases}$$

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We gave lists of equivalent conditions for a representation of M_d to be spatial, and for a representation of L_d to be spatial. Some of them involved the notion of a spatial partial isometry, and some didn't. For example:

- A contractive representation of M_d^p must be spatial.
- If ρ is a representation of L_d such that $\big(\rho(s_1) \;\; \rho(s_2) \;\; \cdots \;\; \rho(s_d)\big)$ is a row isometry, then $\rho(s_j)$ is a spatial isometry for all j.
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- If ρ is a representation of L_d such that $\big(\rho(s_1) \;\; \rho(s_2) \;\; \cdots \;\; \rho(s_d)\big)$ is a row isometry, then $\rho(s_j)$ is a spatial isometry for all j.
- If ρ is a representation of L_d such that $\|\rho(s_j)\|$, $\|\rho(t_j)\| \leq 1$ for all j, and $\rho|_{M_d}$ is contractive as a map on M_d^p, then $\rho(s_j)$ is spatial for all j.

The trickiest part is to prove that if ρ is a contractive representation of M_d^p on $L^p(X, \mu)$, then $\rho(e_{j,j})$ is a multiplication operator.

$t \mapsto \rho\big(1 - e_{j,j} + e^{it} e_{j,j}\big)$ is a homotopy of isometries, spatial since they are surjective. One can check that the associated maps $X \to X$ (or, rather, slightly weaker substitutes) are constant along a homotopy. Therefore they are all the identity map, so $\rho(1 - 2e_{j,j})$ is a multiplication operator.

Question: What happens if we use $\mathbb{R}$ instead of $\mathbb{C}$?

Uniqueness and simplicity

Theorem (Uniqueness)

Let $p \in [1, \infty) \setminus \{2\}$, and let ρ_1 and ρ_2 be spatial representations on L^p-spaces (using σ-finite measures). Then there is an isometric isomorphism $\varphi \colon \overline{\rho_1(L_d)} \to \overline{\rho_2(L_d)}$ such that

$$\varphi(\rho_1(s_j)) = \rho_2(s_j) \quad \text{and} \quad \varphi(\rho_1(t_j)) = \rho_2(t_j)$$

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A representation $\sigma \colon L_d \to L(L^p(Y, \nu))$ is *free* if there is a partition $Y = \coprod_{m \in \mathbb{Z}} E_m$ such that for all $m \in \mathbb{Z}$ and all j, we have

$$\sigma(s_j)(L^p(E_m, \nu)) \subset L^p(E_{m+1}, \nu) \quad \text{and} \quad \sigma(t_j)(L^p(E_m, \nu)) \subset L^p(E_{m-1}, \nu).$$

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Recall: λ is normalized counting measure, $l_d^p = L^p(\{1, 2, \ldots, d\} \, \lambda)$, $X = \{1, 2, \ldots, d\}^{\mathbb{Z}_{>0}}$, and μ is the infinite product of copies of λ.

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Let tr be the normalized trace. Then

$$\frac{1}{\operatorname{card}(G)} \sum_{g \in G} g a g^{-1} = \operatorname{tr}(a) \cdot 1$$

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- Has the property that every contractive unital representation on an L^p space is isometric.

Also, for different values of p, in at least one direction there are no nonzero continuous homomorphisms. So they can't be isomorphic.

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