OPEN PROBLEMS LOOSELY RELATED TO ANALOGS OF CUNTZ ALGEBRAS AND UHF ALGEBRAS ON L^p SPACES

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Abstract. We state a number of problems suggested by recent work on L^p analogs of Cuntz algebras and UHF algebras.

This is a collection of problems suggested by recent work on analogs of Cuntz algebras and UHF algebras acting on L^p spaces. They are grouped in sections of (sometimes loosely) related problems. Some of them, as noted, are already being worked on. Some of them are related to graph algebras, but others are about the general theory of operator algebras on L^p spaces.

1. Generalizations of spatial L^p analogs of Cuntz algebras

Problem 1.1. Extend results for $\mathcal{O}_d^p$ to $\mathcal{O}_\infty^p$. (We do not expect all the equivalent conditions for a representation to be spatial to be still equivalent. But any representation ρ for which $\rho(s_j)$ is a spatial partial isometry with reverse $\rho(t_j)$ for all j should be "good".)

Is $\mathcal{O}_\infty^p$ purely infinite and simple? What is its K-theory?

Problem 1.2. Extend results for $\mathcal{O}_d^p$ to L^p analogs of the extended Cuntz algebras E_d. (The same comment applies as in Problem 1.1.)

Problem 1.3. Suppose that $M_{n_1}(L_{d_1}) \cong M_{n_2}(L_{d_2})$. (This is known to be equivalent to $M_{n_1}(\mathcal{O}_{d_1}) \cong M_{n_2}(\mathcal{O}_{d_2})$.) Does it follow that $M_{n_1}(\mathcal{O}_{d_1}^p) \cong M_{n_2}(\mathcal{O}_{d_2}^p)$? (K-theory shows that the reverse implication holds.)

A particular example to consider, suggested by Gene Abrams, is whether $M_3(\mathcal{O}_5^p)$ is isomorphic to $\mathcal{O}_5^p$. In this case, the isomorphisms for Leavitt algebras and C*-algebras don't send the standard generators to single words in the standard generators, which has the potential to cause problems with norms in the setting of L^p operator algebras.

Problem 1.4. (This problem is being worked on by a graduate student, María Eugenia Rodríguez.) Extend results for $\mathcal{O}_d^p$ to L^p analogs of graph algebra, or at least subclasses (such as algebras of finite graphs, perhaps with no sources or no sinks, or such as Cuntz-Krieger algebras).

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Problem 1.5. This problem is a followup to Problem 1.4. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(This matrix has been called "$\mathbf{2}_-$".) Then $\mathcal{O}_2 \cong \mathcal{O}_A$ but the sign of the determinant flow equivalence invariant is opposite to that for the matrix corresponding to $\mathcal{O}_2$, namely $\det(1 - A^{\mathrm{t}}) > 0$. Is $\mathcal{O}_A^p$ isomorphic to $\mathcal{O}_2^p$?

Problem 1.6. This is also a followup to Problem 1.4. Suppose E and F are graphs (perhaps in a suitable subclass), and $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$. Does it follow that the spatial L^p analogs of these algebras (providing they can be defined) are isomorphic?

Problem 1.7. The space l^p embeds in $\mathcal{O}_\infty^p \subset L(L^p(X, \mu))$ in a very nice algebraic manner: as the closed linear span of the generating isometries. This was used to show that if $\varphi \colon \mathcal{O}_\infty^p \to L(E)$ is a continuous homomorphism, then E contains a subspace isomorphic to l^p. In particular, this is true whenever there is a continuous homomorphism from $L(L^p(X, \mu))$ to $L(E)$.

Can one embed $L^p([0, 1])$ in a similar way in a subalgebra of $L(L^p([0, 1]))$? Does it then follow that if $\psi \colon L(L^p([0, 1])) \to L(E)$ is a continuous homomorphism, then E contains a subspace isomorphic to $L^p([0, 1])$? When $p = 2$, there is such a construction, using creation and annihilation operators on the Fock space (actually, for any separable Hilbert space). However, one gets nothing new, since every separable Hilbert space is isometrically isomorphic to l^2. Presumably when $p \neq 2$, if this construction works (it might not be too hard to imitate the Fock space construction), one gets different algebras.

Problem 1.8. Let $p \in (1, \infty) \setminus \{2\}$, and let $s \in L(l^p(\mathbb{Z}_{>0}))$ be the unilateral shift. Let T_p be the norm closed subalgebra of $L(l^p(\mathbb{Z}_{>0}))$ generated by s and its reverse, the backwards shift. This algebra contains $K(l^p(\mathbb{Z}_{>0}))$. What is the quotient? It is a commutative unital Banach algebra generated by an invertible element and its inverse. Is it isomorphic to the closed subalgebra of $L(l^p(\mathbb{Z}))$ generated by the bilateral shift and its inverse? Is the maximal ideal space isomorphic to S^1? Which functions on the maximal ideal space are in the algebra?

(Something, but not much, is known about the closed subalgebra of $L(l^p(\mathbb{Z}))$ generated by the bilateral shift and its inverse.)

2. Nonspatial representations and representations on other Banach spaces

Problem 2.1. What sort of algebras does one get as $\overline{\rho(L_d)}$ for representations $\rho \colon L_d \to L(L^p(X, \mu))$ which are not spatial? Are they simple? Are they purely infinite? Are they amenable? Do they they have the same K-theory?

Problem 2.2. The spatial L^p Cuntz algebras seem to be "minimal" in some sense. Is there something that deserves to be called a "maximal" L^p analog of $\mathcal{O}_d$? If there is, what can one say about it? In particular, what about the questions in Problem 2.1?

Problem 2.3. What are the right spaces to choose when $p = \infty$? One could consider representations on any of $c_0 = C_0(\mathbb{Z}_{>0})$, l^∞, $L^\infty(X, \mu)$ for a σ-finite measure space $(X, \mathcal{B}, \mu)$, or $C(X)$ for a compact metric spaces X. (Lamperti's Theorem fails for $p = \infty$, so the resulting theory might be quite different.)

Problem 2.4. Develop the theory for representations of L_d on more general Banach spaces than $L^p(X, \mu)$. Which Banach spaces admit representations of L_d? Which ones admit representations ρ which are contractive on generators, strongly forward isometric, or such that both ρ and ρ' are strongly forward isometric? (Some of these answers are probably known. Certainly not all separable Banach spaces admit representations at all. There is a recently constructed infinite dimensional separable Banach space on which every bounded linear map is of the form scalar plus compact operator.) Can one find representations of L_d on nonisomorphic Banach spaces such that the closures of the ranges are nevertheless isomorphic (isometrically isomorphic) as Banach algebras? Can one find a representation $\rho \colon L_d \to L(E)$ such that $\overline{\rho(L_d)}$ is not simple? Can one find one such that $\overline{\rho(L_d)}$ has the "wrong" K-theory? Which Banach spaces E admit representations $\rho \colon L_d \to L(E)$ such that $\overline{\rho(L_d)}$ is simple, or has the "right" K-theory?

The following condition on a unital representation ρ of L_d on a Banach space E seems to be something one might want to require for ρ to be "reasonable".

Definition 2.5. Let E be a Banach space, and let $\rho \colon L_d \to L(E)$ be a representation. We say that ρ is *strongly forward isometric* if if $\rho(s_j)$ is an isometry for every j and for every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$, the element $\rho\left(\sum_{j=1}^{d} \lambda_j s_j\right)$ is a scalar multiple of an isometry.

Problem 2.6. A strongly forward isometric representation ρ of L_d on a Banach space E defines a norm $\|\cdot\|_\rho$ on $\mathbb{C}^d$ by the formula, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$,

$$\left\| \sum_{j=1}^{d} \lambda_j \rho(s_j) \xi \right\| = \|\lambda\|_\rho \|\xi\| \tag{2.1}$$

for all $\xi \in E$. Which norms on $\mathbb{C}^d$ can arise this way? What if one requires that Equation (2.1) hold for all $\lambda \in \mathbb{C}^d$ and $\xi \in E$, but drops the requirement that $\rho(s_j)$ be an isometry for all j? What if one restricts E to lie in some special class of Banach spaces, such as all $L^p(X, \mu)$ for some fixed p? What if one adds the requirement that the representation ρ' also be strongly forward isometric?

Problem 2.7. Fix a norm on $\mathbb{C}^d$ which corresponds to a strongly forward isometric representation ρ as in Problem 2.6. Let σ be some other strongly forward isometric representation of L_d which yields the same norm on $\mathbb{C}^d$. Does it follow that there is an (isometric) isomorphism from $\overline{\rho(L_d)}$ to $\overline{\sigma(L_d)}$ which sends generators to generators? (This is true for the norm corresponding to spatial representations, provided one restricts to L^p spaces of σ-finite measure spaces.)

Problem 2.8. A strongly forward isometric representation ρ of L_∞ (definition similar to Definition 2.5) on a Banach space E defines a norm $\|\cdot\|_\rho$ on $\mathbb{C}^\infty$ by the formula, for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{C}^\infty$,

$$\left\| \sum_{j=1}^{\infty} \lambda_j \rho(s_j) \xi \right\| = \|\lambda\|_\rho \|\xi\|$$

for all $\xi \in E$. Completing $\mathbb{C}^\infty$ in this norm, we get a Banach space of sequences.

First, which Banach spaces can arise this way? What can one say about them?

Second, suppose p is fixed. Consider a representation ρ obtained in the following manner. Let ρ_0 be a spatial representation of L_∞ on l^p (or perhaps a more general strongly forward isometric representation on l^p or on $L^p(X, \mu)$ for some σ-finite measure space $(X, \mathcal{B}, \mu)$), and let $\varphi \colon \overline{\rho_0(L_\infty)} \to L(E)$ be a continuous homomorphism. Then take $\rho = \varphi \circ \rho_0$. Now which Banach spaces can arise? Also, what are the possibilities for E? We know that E must have a subspace isomorphic to l^p. Can anything else be said?

3. L^p analogs of UHF algebras and generalizations

Problem 3.1. For distinct values of p_1 and p_2 in $(1, \infty)$, and arbitrary finite d_1 and d_2, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{(p_1)}$ to $\mathcal{O}_{d_2}^{(p_2)}$. There is always least one direction (from p_1 to p_2 or from p_2 to p_1) in which we know there can be no nonzero continuous homomorphism between spatial L^p UHF algebras. What about the other direction?

Problem 3.2. (This problem is being worked on by Maria Grazia Viola.) Generalize the theory of spatial L^p UHF algebras to spatial L^p AF algebras, including K-theoretic classification, ideal structure, etc.

Problem 3.3. For $p \neq 2$, there are representations of M_d^p on $L^p(X, \mu)$ which are contractive on the standard matrix units but are not isometric. Allowing these presumably gives more general L^p UHF algebras than the spatial ones. If we require that the maps in the direct system still be contractive, do we actually get new algebras? If so, the K-theory must be the same, but are the algebras still simple? (This might follow quickly from results already proved.) Are they isomorphic, or isometrically isomorphic, to the spatial ones?

Problem 3.4. Study analogs of L^p UHF algebras on other Banach spaces or other families of Banach spaces. If $(e_{j,k})_{j,k=1}^d$ is the standard system of matrix units in M_d, then the natural condition on a representation $\rho \colon M_d \to L(E)$ seems, at first sight, to be that $\|\rho(e_{j,k})\| \leq 1$ for all j and k. When can one construct direct systems as for the UHF algebras (with or without this norm condition) such that all the maps in the system are contractive, or all the maps in the system are isometric? The K-theory of the resulting algebras must always be as expected, but when are they simple? When are two of them isomorphic?

Problem 3.5. For $p \in (1, \infty)$, the algebra $\mathcal{O}_d^p$ is a corner in an L^p operator reduced crossed product by $\mathbb{Z}$ of the tensor product of a spatial L^p UHF algebra and $K(l^p)$. (One uses something a bit different from $K(l^p)$ when $p = 1$.) What if one uses a nonspatial L^p UHF algebra? Are there nonspatial UHF algebras on which suitable actions exist? For which representations ρ of L_d can one get $\overline{\rho(L_d)}$ as a crossed product in an analogous way?

4. Representation theory

Problem 4.1. Is there a useful coarse classification of representations of L_d on spaces $L^p(X, \mu)$ which are contractive on generators? Can every representation of L_d on $L^p(X, \mu)$ be decomposed in some way into strongly forward isometric representations? A direct sum decomposition is surely too much to hope for, but what about some kind of direct integral decomposition?

Perhaps one should restrict to strongly forward isometric representations to begin with. Can one usefully divide them into any more classes than spatial and nonspatial?

Problem 4.2. Classify isometric representations of $\mathcal{O}_d^p$ on spaces of the form $L^p(X, \mu)$ up to a suitable equivalence relation. One should bear in mind that, since $\mathcal{O}_d$ is a C*-algebra not of type I, the classification of its representations up to unitary equivalence is generally regarded as hopeless. There are several features for $p \neq 2$ which might lead to a different outcome. First, as shown by Lamperti's Theorem, there are many fewer isometries on $L^p(X, \mu)$ for $p \neq 2$, and hence presumably fewer representations. On the other hand, this also makes it much more difficult for representations to be isometrically equivalent. Furthermore, even the spaces on which we are representing the algebras are not all isometrically isomorphic.

One possibility is to ask for a classification only up to approximate isometric equivalence. For $p = 2$, any two representations on a separable Hilbert space are approximately unitarily equivalent, by Voiculescu's Theorem. But the analog might not be true for $p \neq 2$. Another possibility is to restrict to representations which are free or approximately free. A third possibility is to consider only representations which are part of a family which varies continuously with p, giving perhaps a representation of a continuous field of the Banach algebras $\mathcal{O}_d^p$ on a continuous field of the Banach spaces $L^p(X, \mu)$. (We have not checked that one even gets a continuous field in either case, although it seems likely.)

5. Tensor products and operator space structure

Problem 5.1. We have defined a spatial L^p operator tensor product $A \otimes_p B$ of L^p operator algebras $A \subset L(L^p(X, \mu))$ and $B \subset L(L^p(Y, \nu))$, when μ and ν are σ-finite. Even for norm closed but nonselfadjoint algebras of operators on Hilbert spaces, the tensor product depends on how the algebras are represented.

Is there a canonical choice of this tensor product? (It should be functorial.) Is the spatial tensor product independent of the representations if A and B are equipped with, in addition, systems of matrix norms making them L^p operator spaces in a manner compatible with the algebra multiplication, and the representations are required to be completely isometric?

Is there a useful L^p analog of the maximal tensor product of C*-algebras? If so, are there any general conditions under which the spatial and maximal tensor products are the same? (It would be really great if this followed from Banach algebra amenability of both factors. Such a result is probably much too much to hope for. It can't follow from Banach algebra amenability of just one factor.)

Problem 5.2. Let $d_1, d_2 \in \{2, 3, \ldots\}$. Is the L^p tensor product $\mathcal{O}_{d_1}^p \otimes_p \mathcal{O}_{d_2}^p$ simple? For $d \in \{2, 3, \ldots\}$, is the L^p tensor product of $\mathcal{O}_{d_1}^p$ with a spatial L^p UHF algebra simple?

Are these tensor products purely infinite?

Do the answers depend on the representations used? (See Problem 5.1.)

Problem 5.3. We assume that tensor products of the type considered in Problem 5.2 are in fact simple and products purely infinite. Elliott's Theorem asserts that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. More generally, the classification of purely infinite simple separable nuclear C*-algebras satisfying the Universal Coefficient Theorem provides many isomorphisms of tensor products of Cuntz algebras and also of their tensor products with certain other algebras. For example, $\mathcal{O}_3 \otimes \mathcal{O}_4 \cong \mathcal{O}_2$. (However, $\mathcal{O}_3 \otimes \mathcal{O}_3$ is not a Cuntz algebra.) If D is the UHF algebra of type 2^∞, then $D \otimes \mathcal{O}_2$ and $D \otimes \mathcal{O}_3$ are both isomorphic to $\mathcal{O}_2$, but $D \otimes \mathcal{O}_4 \cong \mathcal{O}_4$.

None of these isomorphisms is valid in the purely algebraic situation. It is known that the tensor product of two Leavitt algebras is never a Leavitt algebra. Leavitt algebras are finitely generated, while the tensor product of one of them with the algebraic analog of the UHF algebra of type 2^∞ is never finitely generated, so this kind of tensor product can also never be a Leavitt algebra.

What happens for the L^p analogs of these algebras? One can first test by computing the K-theory. For C*-algebras, one uses normally uses the Künneth formula, but more primitive methods are available for the examples given. Presumably one gets the same K-theory as in the C*-algebra case. Isomorphism of the algebras, however, seems much more problematic.

Problem 5.4. Is there an abstract characterization of Banach algebras which are isometrically (or isomorphically) representable on a Banach space of the form $L^p(X, \mu)$? Or is there an abstract characterization of a particular class of such algebras? A characterization might involve extra structure. For example, if $A \subset L(L^p(X, \mu))$, then $M_n(A) \subset L\big(L^p\big(\{1, 2, \ldots, n\} \times X\big)\big)$, so A has an L^p analog of operator space structure.

One would hope to prove that if $A \subset L(L^p(X, \mu))$ and if $I \subset A$ is a closed ideal, and possibly assuming extra conditions, than A/I can be isometrically (or isomorphically) represented on $L(L^p(Y, \nu))$ for some Y and ν. This seems hard. A much weaker theorem is known for L^p operator spaces, but nothing resembling the statement we have in mind seems to be known. It may well not be true.

6. L^p operator crossed products

In the following problems, $F^p(G, A, \alpha)$ is the full L^p operator crossed product and $F^p_{\mathrm{r}}(G, A, \alpha)$ is the reduced L^p operator crossed product.

Problem 6.1. (This problem is being worked on by a graduate student, Sanaz Pooya.) Let $p \in [1, \infty) \setminus \{2\}$, let G be a locally compact group, and let (G, A, α) be an isometric G-L^p operator algebra. Is the map

$$\kappa_{\mathrm{r}} \colon F^p(G, A, \alpha) \to F^p_{\mathrm{r}}(G, A, \alpha)$$

necessarily surjective? If G is amenable, is this map necessarily injective? Surjective? Isometric? (In any of these questions, does it help to assume that G is discrete, $G = \mathbb{Z}$, or $A = \mathbb{C}$?) If G is finite, does it follow that κ_{r} is isometric? (It is known that κ_{r} is bijective, but not that it is isometric.)

Problem 6.2. Let X be a compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Define $\alpha \in \mathrm{Aut}(C(X))$ by $\alpha(f) = f \circ h^{-1}$ for $f \in C(X)$. As in the C* case, abbreviate $F^p(\mathbb{Z}, C(X), \alpha)$ to $F^p(\mathbb{Z}, X, h)$ and $F^p_{\mathrm{r}}(\mathbb{Z}, C(X), \alpha)$ to $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$.

Is $F^p(\mathbb{Z}, X, h)$ simple? (The algebra $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ is known to be simple, and we do not know whether it is different from $F^p(\mathbb{Z}, X, h)$.) Can there ever be a nonzero continuous homomorphism from $F^{p_1}(\mathbb{Z}, X_1, h_1)$ to $F^{p_2}(\mathbb{Z}, X_2, h_2)$ or to $F^{p_2}_{\mathrm{r}}(\mathbb{Z}, X_2, h_2)$, or from $F^{p_1}_{\mathrm{r}}(\mathbb{Z}, X_1, h_1)$ to $F^{p_2}_{\mathrm{r}}(\mathbb{Z}, X_2, h_2)$, with $p_1 \neq p_2$ and h_1 and h_2 both minimal?

Problem 6.3. Let $h\colon X \to X$, $F^p(\mathbb{Z}, X, h)$, and $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ be as in Question 6.2. What information about h can one recover from the isomorphism class or isometric isomorphism class of $F^p(\mathbb{Z}, X, h)$ and $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$? For example, if X is the Cantor set, and $h_1, h_2\colon X \to X$ are minimal homeomorphisms, then $C^*(\mathbb{Z}, X, h_1) \cong C^*(\mathbb{Z}, X, h_2)$ if and only if h_1 and h_2 are strongly orbit equivalent. (This is the Giordano-Putnam-Skau Theorem.) But for minimal homeomorphisms $h_1\colon X_1 \to X_1$ and $h_2\colon X_2 \to X_2$, in which both X_1 and X_2 are compact manifolds of dimension at least 2, it seems to be easy to have $C^*(\mathbb{Z}, X_1, h_1) \cong C^*(\mathbb{Z}, X_2, h_2)$ when the dynamics of h_1 and h_2 are quite different, and even when X_1 and X_2 are quite different.

One expects it to be less likely that, for example, $F^p_{\mathrm{r}}(\mathbb{Z}, X_1, h_1) \cong F^p_{\mathrm{r}}(\mathbb{Z}, X_2, h_2)$ than $C^*(\mathbb{Z}, X_1, h_1) \cong C^*(\mathbb{Z}, X_2, h_2)$, since L^p operator algebras are apparently more rigid than C*-algebras.

Problem 6.4. Let $h\colon X \to X$, $F^p(\mathbb{Z}, X, h)$, and $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ be as in Question 6.2. Suppose X is the Cantor set. Does it follow that the invertible elements of $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ are dense? That is, does $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ have stable rank one? This is known for $p = 2$. If $X = S^1$ and h is an irrational rotation, does it follow that the invertible elements of $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ are dense? This is also known for $p = 2$.

In the case $p = 2$, stable rank one holds much more generally. For $p = 2$, both the special examples in Problem 6.4 also have real rank zero. It is known that a unital C*-algebra has real rank zero if and only if it is an exchange ring, and the definition of an exchange ring makes sense for general unital rings. So it seems reasonable to ask the following:

Problem 6.5. In the examples of Question 6.4, is $F^p_{\mathrm{r}}(\mathbb{Z}, X, h)$ an exchange ring?

Problem 6.6. Let $p \in [1, \infty)$. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an isometric action of a locally compact abelian group on an L^p operator algebra. We have constructed dual actions

$$\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}\big(F^p(G, A, \alpha)\big) \quad \text{and} \quad \widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}\big(F^p_{\mathrm{r}}(G, A, \alpha)\big).$$

Is there an analog of Takai duality for the crossed products by these actions?

Problem 6.7. Let $n \in \{2, 3, \ldots\}$, and let $p \in [1, \infty)$. Let A be the reduced L^p operator algebra of the free group on n generators. Is the invertible group of A dense in A? That is, does A have stable rank one? This is known to be true when $p = 2$.

7. Miscellaneous problems

Problem 7.1. What happens to $\mathcal{O}^p_d$ with real scalars? (The K-theory will be different, but this already happens when $p = 2$. We used complex scalars at one crucial place in the proof of equivalence of many of the conditions for a representation to be spatial. Are the results really different?)

Problem 7.2. The algebras $\mathcal{O}^p_d$ are purely infinite and simple. We defined a simple unital Banach algebra to be purely infinite if for every $x \in A \setminus \{0\}$ there exist $x, y \in A$ such that $xay = 1$. How many of the consequences of pure infiniteness of C*-algebras carry over to this situation? Is $K_0(A)$ isomorphic to the set of Murray-von Neumann equivalence classes of nonzero idempotents in A? Is $K_1(A)$ isomorphic to $\mathrm{inv}(A)/\mathrm{inv}_0(A)$? As a special case: Is the invertible group of $\mathcal{O}^p_d$ connected?

Problem 7.3. Our nonisomorphism results are all results on nonisomorphism as Banach algebras. What are the Banach space isomorphism and isometry classifications of the closures of the ranges of representations of Leavitt algebras on L^p spaces? (For $\frac{1}{p} + \frac{1}{q} = 1$, the algebras $\mathcal{O}_d^p$ and $\mathcal{O}_d^q$ are isometrically antiisomorphic.)

Problem 7.4. Suppose $1 \leq p_1 \leq p_2 \leq p_3 < \infty$. Can one get $\mathcal{O}_d^{p_2}$ from $\mathcal{O}_d^{p_1}$ and $\mathcal{O}_d^{p_3}$ by Banach space interpolation?

One should probably start with M_d^p. The question is then whether the norm on $M_d^{p_2}$ can be gotten from those on $M_d^{p_1}$ and $M_d^{p_3}$ by Banach space interpolation. Here one can allow $p = \infty$.

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