## ADDENDUM

What distinguishes some athletes is not so much their technical skills as their intuitive understanding of their sport. In baseball, they are the fielders who always get a jump on the ball; in basketball, they are the players who have "court sense." Similarly, the ability to prove theorems is founded as much on developing mathematical intuition as it is on practicing a large array of technical skills. So we pause here to consider how you can begin developing your own mathematical presence of mind. (Our first advice is to seek an expert second opinion and we especially recommend George Pólya's *How to Solve It* and *Mathematical Discoveries I, II,* Paul Halmos's *How to Write Mathematics,* and Jacques Hadamard's, *An Essay on the Psychology of Invention in the Mathematical Field.*)

There are three states of being unable to prove a theorem.

1. You haven't a clue. You see no reason to believe the result, beyond someone else's assurances that it might be so, and you have no idea how to approach the problem.

The conscious mind does its best to dominate the subconscious because it realizes that the subconscious is not very good at making rational decisions such as deciding to get you out of the path of a Mack truck. Although conscious effort is required to prove a theorem, it is often the subconscious that provides the key, so you should look for ways to free your subconscious. Try working hard on the proof and then, as the saying goes, "sleep on it." Drawing a picture will sometimes trick your conscious mind into dropping its guard, and as we shall see in the next chapter, Venn diagrams show that even something so abstract as de Morgan's laws for logic have an associated picture. Turn the problem upside down. Look for a counterexample, and when (presumably) you can't find one, try to figure out why not. Take away one or more hypotheses until you can find an example. Then think how the hypothesis you have deleted can be used in a proof. Can you translate the problem into a new setting? For example, what makes analytic geometry an important part of calculus is that the cartesian coordinate system is essentially a device for translating back and forth between geometry and algebra. Can you relate the problem to some other result you already know? Is there some consequence of the result that you can prove? Sometimes, when you work out part of a result, you will see how the rest follows. And then there is Pólya's advice, "See if you can think of the result as part of a result that comprises it."

2. You feel certain that the result is true and you think there is a natural approach, but when you make the argument that seems headed in the right direction you get stuck.

Go back to make sure that you have used all the hypotheses. If so, have you given up too much ground? For example, if one hypothesis is that $x > 5$ and all you use is that $x \neq -1$, then you are probably leaving too much of this hypothesis unused. Obviously you should use all the available information. If after an honest effort you cannot use some part of the hypothesis, you should then consider a contrapositive proof or a proof by contradiction. Such a proof allows you to write down what appears to be a new piece of information and thereby gives you a new starting point.

3. You know the result is true because you have thought of an argument that establishes it. Unfortunately, your argument does not persuade a knowledgeable and patient reader such as your teacher or the brightest of your classmates.

EXERC

For most people this state is not so common as states (1) and (2) and is not really so important a concern. But if you encounter state (3) repeatedly, it is a sign that you are allowing yourself to gloss over gaps in your argument or that you are allowing yourself to write up your results in your own private language as if you were writing a diary rather than a communication. One way of encoding your proof so that it cannot be deciphered, which is particularly irksome, is to use a letter or symbol without disclosing its meaning. You should treat your readers as royalty by introducing each symbol you use into their gracious presence.

Here is how to check for gaps. Look at the last statement first. Is it what you wanted to prove? If your conclusion is that $1 = 1$, you may very well have a sound proof of this equality, but surely you do not have a proof of the proposition you wish to establish. (You will see a subtler example of this gap illustrated by Exercise 31 of Section 3.2.) Once you know that your last conclusion is the result you want, go back to check the first line of your proof. Make sure that it follows from the hypotheses and that it is relevant. Thereafter, it is just a matter of applying a simple principle—if there is going to be a mistake in your proof there has got to be a first mistake. That's the one to avoid. Similarly, if a classmate or teacher cannot follow your argument, you are entitled to know the first time your explanation is deemed unsatisfactory. When looking for the gap in your argument, you should be especially wary of a step that appears obvious. Thus you should view the words *obvious* and *obviously* as warning signs that the step in question is suspect.

We have already given the most important advice about writing up mathematical arguments clearly: read Halmos's *How to Write Mathematics*. There are, however, a few simple tricks worth mentioning here. Watch out for the quantifiers *for each* and *there is*, and watch out for the little words like *the*, *all*, *and*, *or*, and *only*. Don't abuse notation or terminology. For example, respect the distinction between $\in$ and $\subseteq$ and don't use $=$ as if this mathematical symbol were just shorthand—"$A - B \in C \cap B' = B \cup C' \subseteq B \cup A'$" is gibberish.

We have come to the end of our advice, but we pass along invaluable advice given us by Professor William Jenner.

**Jenner's Law: If after working long into the night you think you have discovered a proof, write down the idea and without scrutinizing the argument go to bed.**

If your proof is correct, you will have a better chance of writing up a clear presentation of it in the morning. If the proof is incorrect, you will have maximized the joy and excitement of mathematical discovery, and in the morning the mistake will be obvious. Besides, sometimes your conscious mind will get so distracted by all the excitement that it will allow your subconscious to tell you the next morning both where the mistake is and how to fix it.