Math 5350/6350 Functional Analysis — Exam 1

The following problems are **due at the beginning of class on Wednesday, September 29**. Please follow all guidelines as described in the "Homework" section of the course syllabus. For this exam you are allowed to use your notes and the class textbook as well as ask questions of the instructor. You are **not** allowed to use outside books or the internet, and you are **not** allowed to talk to anyone other than the instructor about the problems or their solutions.

1. (a) Let $(X, \|\cdot\|)$ be a normed space. Prove that the function $x \mapsto \|x\|$ is a continuous function from X to $\mathbb{R}$.

   (b) Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Prove that for any fixed $y \in X$ the function $x \mapsto \langle x, y\rangle$ is a continuous function from X to $\mathbb{C}$. Also prove that for any fixed $y \in X$ the function $x \mapsto \langle y, x\rangle$ is a continuous function from X to $\mathbb{C}$.

2. Recall that a topological space is called *separable* if it contains a countable dense subset.

   (a) Prove that for any $p \in [1, \infty)$ the Banach space ℓ^p is separable.

   (b) Let

   $$\ell^\infty := \{(z_1, z_2, \ldots) \in \prod_{n=1}^{\infty} \mathbb{C} : \exists M \in \mathbb{R} \text{ such that } |z_n| < M \text{ for all } n \in \mathbb{N}\}$$

   with the norm $\|(z_1, z_2, \ldots)\|_\infty := \sup\{|z_n| : n \in \mathbb{N}\}$. Prove that the Banach space ℓ^∞ is not separable.

3. Consider

   $$\bigoplus_{n=1}^{\infty} \mathbb{C} := \{(x_1, x_2, \ldots) : x_i \in \mathbb{C} \text{ and only finitely many } x_i \text{ are nonzero}\}.$$

   Prove that there is no norm $\|\cdot\|$ on $\bigoplus_{n=1}^{\infty} \mathbb{C}$ such that $(\bigoplus_{n=1}^{\infty} \mathbb{C}, \|\cdot\|)$ is complete.

4. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of real numbers with the following property: whenever we have a sequence $\{\beta_n\}_{n=1}^{\infty}$ of real numbers converging to 0, the infinite sum $\sum_{n=1}^{\infty} \alpha_n \beta_n$ is convergent. Prove $\sum_{n=1}^{\infty} |\alpha_n| < \infty$. (Hint: Use the Principle of Uniform Boundedness.)

5. Let $(X, \|\cdot\|)$ be a a normed space over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). Prove that if $\vec{u}, \vec{v} \in \operatorname{Ball} X$ are norm one vectors such that neither $\vec{u}$ nor $\vec{v}$ is a scalar multiple of the other, then there exists a linear functional $f : X \to \mathbb{F}$ with $\|f\| = 1$, $f(\vec{u}) = 1$, and $f(\vec{v}) = 0$.

A Note Regarding the Write-Up of Proofs: When writing proofs, you will often need to first do work on scratch paper and then write up your final solution once you have figured out how to do the proof. You should only hand in the final result — not the scratchwork. The write-up that you hand in should be written neatly and legibly. Also, when you are writing proofs remember that the standard rules of English usage still apply; in particular, you should use complete sentences and proper punctuation. As in any class that requires writing assignments, your grade will be based in part on your ability to write clearly, convincingly, and correctly.

When writing up proofs, you should write out the statement you are proving followed by your proof. For example, if a problem asked you to show that A and B imply C, then you should write

Claim: If A and B, then C.

Proof: Since A and B hold we see that . . .

$\square$

By doing this, you begin with a statement of your assumptions and with a carefully worded statement of the result to be proven. This will help you keep track of what you are allowed to assume and what you are trying to show.