Exam 2

The following are due at the beginning of class on Friday, October 31.

Problem 1: (20 points) Let H be a Hilbert space. Prove that if $T \in \mathcal{K}(H)$, then either $\operatorname{rank} T < \infty$ or $\operatorname{Ran}(T)$ is not closed.

Problem 2: If V is a vector space over $\mathbb{C}$, then a *vector space partial order* on V is a relation $\leq$ satisfying the following:

(i) $x \leq x$ for all $x \in V$.

(ii) If $x, y \in V$ with $x \leq y$ and $y \leq x$, then $x = y$.

(iii) If $x, y, z \in V$ with $x \leq y$ and $y \leq z$, then $x \leq z$.

(iv) If $x, y \in V$ and $x \leq y$, then $x + z \leq y + z$ for all $z \in V$.

(v) If $x, y \in V$ and $x \leq y$, then $rx \leq ry$ for all $r \in [0, \infty)$.

Note that properties (i)–(ii) simply say that $\leq$ is a partial order, and properties (iv)–(v) require that partial order to interact appropriately with the addition and scalar multiplication on V.

If V is a vector space over $\mathbb{C}$, a *cone* in V is a subset $C \subseteq V$ satisfying:

(I) If $x, y \in C$, then $x + y \in C$.

(II) If $x \in C$ and $r \in [0, \infty)$, then $rx \in C$.

(III) $C \cap -C = \{0\}$.

(a) (10 points) Prove that if $\leq$ is a vector space partial order, then $C := \{x \in V : x \geq 0\}$ is a cone.

(b) (10 points) Prove that if C is a cone, and if we define a relation $\leq$ on V by $x \leq y$ if and only if $y - x \in C$, then $\leq$ is a vector space partial order on V.

(c) (20 points) Let H be a Hilbert space. Prove that the positive operators in $B(H)$ form a cone. Conclude that we may define a vector space partial ordering on $B(H)$ by: $S \leq T$ if and only if $T - S$ is a positive operator.

(d) (10 points) We already defined a partial ordering $\leq_p$ on the projections as follows: If P, Q are projections on H, then $P \leq_p Q$ if and only if $QP = P$. Prove that this partial ordering on the projections coincides with the partial ordering provided by the positive operators in $B(H)$; that is, prove that if Q and P are projections, then $P \leq_p Q$ if and only if $P \leq Q$. (Feel free to use any results we proved in class.)

Problem 3: If $T : H \to H$ is linear (but not necessarily bounded), we say that T is *orthogonally diagonalizable* if there exists an orthonormal basis $\{e_i\}_{i \in I}$ such that for each $i \in I$ we have $T(e_i) = \lambda_i e_i$ for some $\lambda_i \in \mathbb{C}$.

Suppose that H is a separable infinite-dimensional Hilbert space, that T is orthogonally diagonalizable, and that $\{e_i\}_{i=1}^{\infty}$ is a countably infinite orthonormal basis for H with $T(e_i) = \lambda_i e_i$ for $i \in \mathbb{N}$. Note that with this choice of basis, we may identify H with $\ell^2(\mathbb{N})$ and we may identify T with the diagonal infinite matrix indexed by $\mathbb{N}$ whose diagonal entries are $\lambda_1, \lambda_2, \ldots$.

(a) (10 points) Prove that T is a bounded operator if and only if $\lambda_1, \lambda_2, \ldots$ is a bounded sequence.

(b) (10 points) Prove that T is a compact operator if and only if the sequence $\lambda_1, \lambda_2, \ldots$ has the property that $\lim_{n \to \infty} \lambda_n = 0$.

(c) (10 points) Prove that T is a finite-rank operator if and only if the sequence $\lambda_1, \lambda_2, \ldots$ has only a finite number of nonzero terms.

Note that if we identify the orthogonally diagonalizable on H with the sequence $\{\lambda_n\}_{n=1}^{\infty}$ of eigenvalues (equivalently, diagonal entries), then the bounded operators correspond to $\ell^\infty := L^\infty(\mathbb{N})$, the compact operators correspond to $c_0 := C_0(\mathbb{N})$, and the finite-rank operators correspond to $c_{00} := C_c(\mathbb{N})$.