Math 7320 — Final Exam

The following are due in my mailbox at Noon on Monday, Dec. 15.

Problem 1: If A is a unital C^*-algebra, an element $a \in A$ is *positive* if a is self-adjoint and $\sigma(a) \subseteq [0, \infty)$.

(i) (10 points) Prove that if $a \in A$ is self-adjoint, then there exist positive elements $a_+, a_- \in A$ such that $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.

(ii) (5 points) Prove that the positive elements a_+ and a_- described in part (i) are unique.

(iii) (10 points) If $a \in A$ is self-adjoint, define $|a| := (a^2)^{1/2}$. Prove that $|a| = a_+ + a_-$.

(iv) (5 points) Prove that every element of a C^*-algebra is a linear combination of four positive elements.

Problem 2: (30 points) Let A be a unital C^*-algebra. Prove that for any $a \in A$ the following three conditions are equivalent.

(a) a is positive.

(b) $a = b^2$ for some self-adjoint element $b \in A$.

(c) $a = x^* x$ for some $x \in A$.

Problem 3: (20 points) Prove that if A and B are unital C^*-algebras and $h : A \to B$ is a $*$-homomorphism with $h(1) = 1$, then $\|h(a)\| \leq \|a\|$ for all $a \in A$. Also prove that $\operatorname{im} h$ is closed, and that h is isometric (i.e., $\|h(a)\| = \|a\|$) if and only if h is injective.

Problem 4: Let A be a unital C^*-algebra, let I be a proper (closed, two-sided) ideal in A, and let $a \in I$ be self-adjoint.

(a) (5 points) Prove that $0 \in \sigma(a)$.

(b) (15 points) Prove that if $f : \sigma(a) \to \mathbb{C}$ is a continuous function with $f(0) = 0$, then $f(a) \in I$.