Talk 5: Generalizations of Graph Algebras within the Cuntz-Pimsner Framework

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"Generalizations without details are hollow, but details without generalizations are barren."

The Cuntz-Pimsner algebras provide a vast generalization of graph C^*-algebras, in which some — but not all — of the graph C^*-algebra results extend.

As with all generalizations there is a balance to be struck between finding classes of C^*-algebras that are large enough to contain many examples, and yet specific enough that useful theorems can be proven. And, as always, the more objects we try to talk about at once, the less we can say about them.

It's often useful to consider subclasses of Cuntz-Pimsner algebras that generalize (i.e., include most of) the graph C^*-algebras, in the hopes that more graph C^*-algebra theorems will generalize to this setting.

We will give a survey of some generalizations of graph algebras. For each class we will:

1. define the basic objects that will be used in place of directed graphs, and discuss how a C^*-algebra can be constructed from such an object,
2. explain how graph algebras are special cases of these C^*-algebras, and
3. compare and contrast the theory for these C^*-algebras to the theory for graph C^*-algebras.

We will consider three classes of generalizations:

- Exel-Laca algebras
- ultragraph algebras
- topological graph algebras

One desirable characteristic for our classes is to have a notion of Condition (L), which will give a Cuntz-Krieger Uniqueness Theorem and conditions for simplicity.

Exel-Laca Algebras

Background: The Cuntz-Krieger algebras are C^*-algebras associated to $n \times n$ matrices with entries in $\{0, 1\}$.

Definition

If A is an $n \times n$ matrix with entries in $\{0, 1\}$ and no zero rows, then the *Cuntz-Krieger algebra* $\mathcal{O}_A$ is the universal C^*-algebra generated by a collection of partial isometries $\{S_i : 1 \leq i \leq n\}$ satisfying the relations

$$S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^* \qquad \text{and} \qquad \sum_{i=1}^{n} S_i S_i^* = I.$$

When A has all 1's as entries this is the Cuntz algebra $\mathcal{O}_n$.

Exel-Laca algebras are Cuntz-Krieger algebras for infinite matrices.

Definition (Exel and Laca)

Let I be a countable set and let $A = \{A(i,j)_{i,j \in I}\}$ be a $\{0,1\}$-matrix over I with no identically zero rows. The Exel-Laca algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ with commuting initial projections and mutually orthogonal range projections satisfying

$$s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$$

and

$$\prod_{x \in X} s_x^* s_x \prod_{y \in Y} (1 - s_y^* s_y) = \sum_{j \in I} A(X,Y,j) s_j s_j^* \quad (\dagger)$$

whenever X and Y are finite subsets of I such that the function

$$j \in I \mapsto A(X,Y,j) := \prod_{x \in X} A(x,j) \prod_{y \in Y} (1 - A(y,j))$$

is finitely supported.

To understand the motivation for this last relation comes from, notice that combinations of formal infinite sums obtained from the original Cuntz-Krieger relations could give relations involving finite sums, and (†) says that these finite relations must be satisfied in $\mathcal{O}_A$

Although there is reference to a unit in (†), this relation applies to algebras that are not necessarily unital, with the convention that if a 1 still appears after expanding the product in (†), then the relation implicitly states that $\mathcal{O}_A$ is unital.

If E is a graph with no sinks or sources, then $C^*(E)$ is an Exel-Laca algebra. In fact, it can be shown that if E has no sinks or sources, and if $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family, then $\{s_e : e \in E^1\}$ is a collection of partial isometries satisfying the relations defining $\mathcal{O}_{B_E}$, where B_E is the edge matrix of E.

$$B_E(e,f) = \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{if } r(e) \neq s(f) \end{cases}$$

Not all graph algebras are Exel-Laca algebras; there are examples of graphs with sinks, and other examples of graphs with sources, whose C^*-algebras are not isomorphic to any Exel-Laca algebra.

There is a Cuntz-Krieger Uniqueness Theorem for Exel-Laca algebras. If A is a countable square matrix over I with entries in $\{0, 1\}$, then we define a directed graph $\mathrm{Gr}(A)$, by letting the vertices of this graph be I, and then drawing an edge from i to j if and only if $A(i, j) = 1$.

Theorem (Cuntz-Krieger Uniqueness)

Let I be a countable set and let $A = \{A(i, j)_{i,j \in I}\}$ be a $\{0, 1\}$-matrix over I with no identically zero rows. If $\mathrm{Gr}(A)$ satisfies Condition (L), and if $\rho : \mathcal{O}_A \to B$ is a $*$-homomorphism between C^*-algebras with the property that $\rho(S_i) \neq 0$ for all $i \in I$, then ρ is injective.

The graph Gr(A) is also useful in describing pure infiniteness of Exel-Laca algebras.

Theorem (Exel and Laca)

Every nonzero hereditary subalgebra of $\mathcal{O}_A$ contains an infinite projection if and only if Gr(A) *satisfies Condition (L) and every vertex in* Gr(A) *can reach a cycle in* Gr(A).

Simplicity for Exel-Laca algebras is more complicated. Exel and Laca showed that if $\mathrm{Gr}(A)$ is transitive and not a single cycle, then $\mathcal{O}_A$ is simple.

A complete characterization of simplicity was obtained by Szymański; he defined a notion of saturated hereditary subset for A, and proved that $\mathcal{O}_A$ is simple if and only if $\mathrm{Gr}(A)$ satisfies Condition (L) and A has no proper nontrivial saturated hereditary subsets.

Note: There are examples of a matrix A such that $\mathcal{O}_A$ is simple, but $C^*(\mathrm{Gr}(A))$ is not simple!

Szymański's result can also be used to show that the dichotomy holds for simple Exel-Laca algebras: every simple Exel-Laca algebra is either AF or purely infinite.

In addition, the universal property of $\mathcal{O}_A$ gives a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_A$ with $\gamma_z(S_i) = zS_i$, and there is a gauge-invariant uniqueness theorem for Exel-Laca algebras.

Exel and Laca have also calculated the K-theory of $\mathcal{O}_A$, as $K_0(\mathcal{O}_A) \cong \operatorname{coker}(A^t - I)$ and $K_1(\mathcal{O}_A) \cong \ker(A^t - I)$, where $A^t - I : \bigoplus_I \mathbb{Z} \to \mathcal{R}$ and $\mathcal{R}$ is an appropriate codomain.

Ultragraph Algebras

Exel-Laca algebra lack some of the visual appeal found with graph algebras. There has been an attempt to study Exel-Laca algebras using a generalized notion of a graph, called an "ultragraph".

An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to P(G^0)$, where $P(G^0)$ denotes the collection of nonempty subsets of G^0.

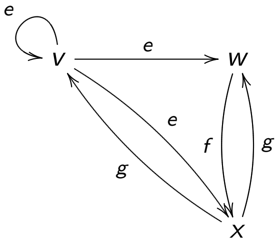
Note that a graph may be viewed as a special type of ultragraph in which $r(e)$ is a singleton set for each edge e.

A convenient way to draw ultragraphs is to first draw the set G^0 of vertices, and then for each edge $e \in \mathcal{G}^1$ draw an arrow labeled e from $s(e)$ to each vertex in $r(e)$. For instance, the ultragraph given by

$$G^0 = \{v, w, x\} \quad \text{and} \quad \mathcal{G}^1 = \{e, f, g\}$$

$$s(e) = v \quad s(f) = w \quad s(g) = x$$
$$r(e) = \{v, w, x\} \quad r(f) = \{x\} \quad r(g) = \{v, w\}$$

may be drawn as



Thus in the above example there are only three edges, e, f, and g, despite the fact that there are six arrows drawn.

For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ we let $\mathcal{G}^0$ denote the smallest subcollection of the power set of G^0 that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections, finite unions, and relative complements (i.e., $A, B \in \mathcal{G}^0$ implies $A \setminus B \in \mathcal{G}^0$).

Definition (T)

If $\mathcal{G}$ is an ultragraph, a *Cuntz-Krieger $\mathcal{G}$-family* is a collection of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a collection of projections $\{p_A : A \in \mathcal{G}^0\}$ that satisfy

1. $p_\emptyset = 0$, $p_A p_B = p_{A \cap B}$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ for all $A, B \in \mathcal{G}^0$
2. $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$
3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ when $0 < |s^{-1}(v)| < \infty$.

We define $C^*(\mathcal{G})$ to be the C^*-algebra generated by a universal Cuntz-Krieger $\mathcal{G}$-family.

When A is a singleton set $\{v\}$, we write p_v in place of $p_{\{v\}}$.

When $\mathcal{G}$ has the property that $r(e)$ is a singleton set for every edge e, then $\mathcal{G}$ may be viewed as a graph (and, in fact, every graph arises this way).

In this case $\mathcal{G}^0$ is simply the finite subsets of G^0, and if $\{s_e, p_v\}$ is a Cuntz-Krieger family for the graph algebra associated to $\mathcal{G}$, then by defining

$$p_A := \sum_{v \in A} p_v$$

we see that $\{p_A, s_e\}$ is a Cuntz-Krieger $\mathcal{G}$-family.

Thus the graph algebra and the ultragraph algebra for $\mathcal{G}$ coincide in this situation.

Moreover, every Exel-Laca algebra is an ultragraph algebra.

Given a matrix B, one creates an ultragraph $\mathcal{G}_B$ with edge matrix B, and then $C^*(\mathcal{G}_B) \cong \mathcal{O}_B$.

Thus ultragraphs give a framework for studying graph algebras and Exel-Laca algebras simultaneously.

A *path* in an ultragraph $\mathcal{G}$ is a sequence of edges $\alpha_1 \ldots \alpha_n$ with $s(\alpha_i) \in r(\alpha_{i-1})$ for $i = 2, 3, \ldots, n$

Definition

If $\mathcal{G}$ is an ultragraph, then a *cycle* is a path $\alpha_1 \ldots \alpha_n$ with $s(\alpha_1) \in r(\alpha_n)$. An *exit* for a cycle is either of the following:

1. an edge $e \in \mathcal{G}^1$ such that there exists an i for which $s(e) \in r(\alpha_i)$ but $e \neq \alpha_{i+1}$
2. a sink w such that $w \in r(\alpha_i)$ for some i.

(Note that if $\alpha_1 \ldots \alpha_n$ is a cycle without an exit, then $r(\alpha_i)$ is a single vertex for all i.)

Condition (L): Every cycle in $\mathcal{G}$ has an exit; that is, for any cycle $\alpha := \alpha_1 \ldots \alpha_n$ there is either an edge $e \in \mathcal{G}^1$ such that $s(e) \in r(\alpha_i)$ and $e \neq \alpha_{i+1}$ for some i, or there is a sink w with $w \in r(\alpha_i)$ for some i.

Theorem (Cuntz-Krieger Uniqueness)

Let $\mathcal{G}$ be an ultragraph satisfying Condition (L). If $\rho : C^*(\mathcal{G}) \to B$ is a $*$-homomorphism between C^*-algebras, and if $\rho(p_v) \neq 0$ for all $v \in G^0$, then ρ is injective.

Note that if $\rho(p_v) \neq 0$ for all $v \in G^0$, then $\rho(p_A) \neq 0$ for all nonempty $A \in \mathcal{G}^0$, since p_A dominates p_v for all $v \in A$.

Furthermore, by the universal property for $C^*(\mathcal{G})$ there exists a gauge action $\gamma_z : \mathbb{T} \to \text{Aut}\, C^*(\mathcal{G})$ with $\gamma_z(p_A) = p_A$ and $\gamma_z(s_e) = zs_e$ for all $A \in \mathcal{G}^0$ and $e \in G^1$.

Theorem (Gauge-Invariant Uniqueness)

Let $\mathcal{G}$ be an ultragraph, $\{s_e, p_A\}$ the canonical generators in $C^*(\mathcal{G})$, and γ the gauge action on $C^*(\mathcal{G})$. Also let B be a C^*-algebra, and $\rho : C^*(\mathcal{G}) \to B$ be a $*$-homomorphism for which $\rho(p_v) \neq 0$ for all $v \in G^0$. If there exists a strongly continuous action β of $\mathbb{T}$ on B such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$, then ρ is injective.

To discuss simplicity we need a notion of saturated hereditary collections. A subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *hereditary* if

1. whenever e is an edge with $\{s(e)\} \in \mathcal{H}$, then $r(e) \in \mathcal{H}$
2. $A, B \in \mathcal{H}$, implies $A \cup B \in \mathcal{H}$
3. $A \in \mathcal{H}$, $B \in \mathcal{G}^0$, and $B \subseteq A$, imply that $B \in \mathcal{H}$.

A hereditary subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *saturated* if for any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$ we have that

$$\{r(e) : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq \mathcal{H} \Longrightarrow \{v\} \in \mathcal{H}.$$

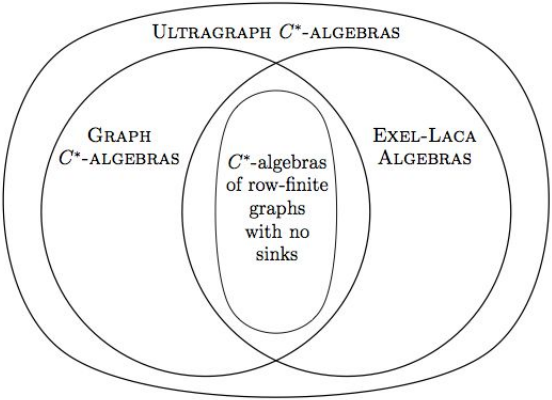
Theorem (T)

An ultragraph algebra $\mathcal{G}$ is simple if and only if $\mathcal{G}$ satisfies Condition (L) and $\mathcal{G}^0$ contains no saturated hereditary subcollections other than $\emptyset$ and $\mathcal{G}^0$.

In addition, the dichotomy holds for simple ultragraph algebras; every simple ultragraph algebra is either AF or purely infinite.

Question: How do ultragraph algebras, Exel-Lacal algebras, and graph algebras compare?

For the isomorphism classes, we get the following diagram.

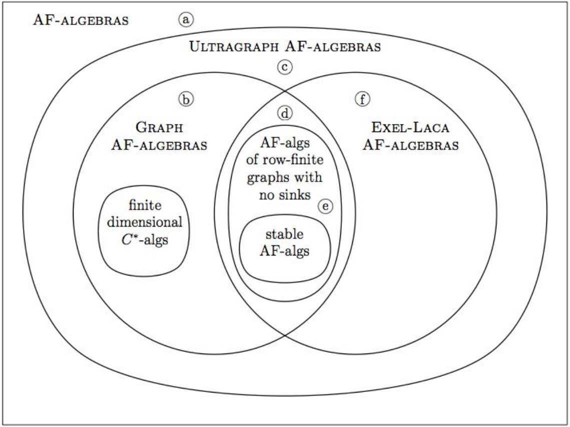


When we only look up to Morita equivalence, there is no difference among the classes.

Theorem (Katsura, Sims, T)

Ultragraph algebras, Exel-Lacal algebras, and graph algebras coincide up to Morita equivalence.

Let's revisit the isomorphism classes, and to get an idea of the differences, let's consider AF-algebras. Each class contains different AF-algebras, and the AF-algebras are sufficient to distinguish among the classes.



Region	unital C^*-algebra	nonunital C^*-algebra
(a)	c_c	$c_0 \oplus c_c$
(b)	$\mathcal{K}^+$	c_0
(c)	$M_{2^\infty} \oplus \mathbb{C}$	$M_{2^\infty} \oplus \mathbb{C} \oplus \mathcal{K}$
(d)	$M_2(\mathcal{K}^+)$	$M_2(\mathcal{K}^+) \oplus \mathcal{K}$
(e)	—	$C^*(F_2)$
(f)	M_{2^∞}	$M_{2^\infty} \oplus \mathcal{K}$

$\mathcal{K}^+$ = unitization of $\mathcal{K}$, $c_0 := \{f : \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} f(n) = 0\}$, $c_c := \{f : \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} f(n) \in \mathbb{C}\}$

F_2 denotes the graph $v_1 \rightrightarrows v_2 \rightrightarrows v_3 \rightrightarrows v_4 \cdots$

Definition (Katsura)

A *topological graph* is a quadruple

$$E = (E^0, E^1, r, s)$$

consisting of a second countable locally compact Hausdorff space E^0 (whose elements are called vertices), a second countable locally compact Hausdorff space E^1 (whose elements are called edges), a local homeomorphism $r : E^1 \to E^0$, and a continuous map $s : E^1 \to E^0$.

(Katsura actually interchanged the roles of r and s. We write it this way for consistency with our other notation.)

Note that when E^0 and E^1 have the discrete topology, this is just a graph.

Topological graphs have been studied extensively by Katsura.

To create the C^*-algebra associated to E, we mimic the construction of the graph C^*-correspondence.

Given a topological graph $E = (E^0, E^1, r, s)$ let $A := C_0(E^0)$ and define an A-valued inner product on $C_c(E^1)$ by

$$\langle \xi, \eta \rangle_A(v) := \sum_{r(\alpha)=v} \overline{\xi(\alpha)}\eta(\alpha).$$

The fact that r is a local homeomorphism assures us that this sum is finite.

We let $X(E)$ denote the closure of $C_c(E^1)$ in the norm arising from this inner product. We define a right action of A on $X(E)$ by

$$\xi \cdot f(\alpha) := \xi(\alpha)f(r(\alpha))$$

and extending to all of X. We also define a left action $\phi : A \to \mathcal{L}(X)$ by setting

$$\phi(f)\xi(\alpha) := f(s(\alpha))\xi(\alpha)$$

and extending to all of $X(E)$. We call $X(E)$ the *C^*-correspondence associated to E.*

Definition (Katsura)

If E is a topological graph, then we define $C^*(E) := \mathcal{O}_{X(E)}$, where $X(E)$ is the C^*-correspondence associated to E. We let (π_E, t_E) denote the universal $J_{X(E)}$-coisometric representation of $X(E)$ into $C^*(E)$.

Since $A := C_0(E^0)$ is a commutative C^*-algebra, ideals of A correspond to open subsets of E^0.

If $E = (E^0, E^1, r, s)$ is a topological graph, we define the following:

1. $E^0_{\text{sinks}} := E^0 \setminus \overline{s(E^1)}$
2. $E^0_{\text{fin}} := \{v \in E^0 :$ there exists a precompact neighborhood V of v such that $s^{-1}(\overline{V})$ is compact. $\}$
3. $E^0_{\text{reg}} := E^0_{\text{fin}} \setminus \overline{E^0_{\text{sinks}}}$

Then

$$\ker \phi = C_0(E^0_{\text{sinks}}), \quad \phi^{-1}(\mathcal{K}(X)) = C_0(E^0_{\text{fin}}), \quad \text{and} \quad J_X = C_0(E^0_{\text{reg}}).$$

Because they are Cuntz-Pimsner algebras, topological graph algebras have a natural gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ with

$$\gamma_z(\pi_E(a)) = \pi_E(a) \quad \text{and} \quad \gamma_z(t_E(x)) = z\, t_E(x)$$

for $a \in A$ and $x \in X(E)$.

Theorem (Guage-Invariant Uniqueness)

Let E be a topological graph. Let $\rho : C^*(E) \to B$ be a $*$-homomorphism between C^*-algebras with the property that $\rho|_{\pi_E(A)}$ is injective. If there exists a gauge action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$, then ρ is injective.

Let $E = (E^0, E^1, r, s)$ be a topological graph. We say that a subset $U \subseteq E^0$ is *hereditary* if whenever $\alpha \in E^1$ and $s(\alpha) \in U$, then $r(\alpha) \in U$. We say that a hereditary subset U is *saturated* if $v \in E^0_{\text{reg}}$ and $r(s^{-1}(v)) \subseteq U$ implies $v \in U$.

Theorem

Let $E = (E^0, E^1, r, s)$ be a topological graph with the property that $E^0_{\text{reg}} = E^0$. Then there is a bijective correspondence from the set of saturated hereditary open subsets of E^0 onto the gauge-invariant ideals of $C^*(E)$ given by

$$U \mapsto \mathcal{I}_U := \text{the ideal in } C^*(E) \text{ generated by } \pi_E(C_0(U)).$$

Furthermore, $\mathcal{I}_U$ is Morita equivalent to $C^*(E_U)$, where E_U is the subgraph of E whose vertices are U and whose edges are $s^{-1}(U)$, and $C^*(E)/\mathcal{I}_U \cong C^*(E \setminus U)$, where $E \setminus U$ is the subgraph of E whose vertices are $E^0 \setminus U$ and edges are $E^1 \setminus r^{-1}(U)$.

(In general, the gauge-invariant ideals of $C^*(E)$ correspond to pairs (U, V) of admissible subsets.)

There is a version of Condition (L) and a Cuntz-Krieger Uniqueness Theorem for topological graph algebras. Note that Condition (L) makes use of the topology on E^0.

Condition (L): The set of base points of cycles in E with no exits has empty interior.

Theorem (Cuntz-Krieger Uniqueness)

Let E be a topological graph that satisfies Condition (L). If $\rho : C^*(E) \to B$ is a $*$-homomorphism from $C^*(E)$ into a C^*-algebra B with the property that the restriction $\rho|_{\pi_E(A)}$ is injective, then ρ is injective.

Furthermore, simplicity of topological graph algebras algebras has been characterized

Theorem

The topological graph algebra $C^*(E)$ is simple if and only if E satisfies Condition (L) and there are no saturated hereditary open subsets of E^0 other than $\emptyset$ and E^0.

The dichotomy does not hold for topological graph algebras: There are simple topological graph algebras that are neither AF nor purely infinite.

Also, there is a version of Condition (K) for topological graph algebras.

We write $w \geq v$ to mean that there is a path α with $s(\alpha) = w$ and $r(\alpha) = v$. We also define

$$v^{\geq} := \{w \in E^0 : w \geq v\}.$$

Condition (K): The set

$$\{v \in E^0 : v \text{ is the base point of exactly one simple cycle and } v \text{ is isolated in } v^{\geq}\}$$

is empty.

Theorem

Let $E = (E^0, E^1, r, s)$ be a topological graph that satisfies Condition (K). Then every ideal in $C^*(E)$ is gauge invariant.

The class of topological graph algebras contains

- every AF algebra
- every Kirchberg algebra (and hence all K-groups are possible)
- Matsumoto's C^*-algebras associated to subshifts
- every ultragraph algebra
- every Exel-Laca algebra
- every graph algebra
- many other classes we haven't talked about

Remark: I believe that currently there are no known examples of a nuclear C^*-algebra satisfying the Universal Coefficients Theorem that is not a topological graph algebra.

Topological graphs can also be used to study ultragrph algebras (which contain Exel-Laca algebras and graph C^*-algebras as special cases).

Given an ultragraph $\mathcal{G}$, one can build a topological graph $E_{\mathcal{G}}$ in such a way that the ultragraph C^*-algebra $C^*(\mathcal{G})$ and the topological graph algebra $C^*(E_{\mathcal{G}})$coincide. Topological graph algebra results can then be used to study ultragraph algebras, and give theorems that have not been obtained by any other methods.