Extensions of graph C^*-algebras

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Abstract

We consider C^*-algebras associated to row-finite (directed) graphs and examine the effect that adding a sink to the graph has on the associated C^*-algebra. In Chapter 2 we give a precise definition of how a sink may be added to a graph, and discuss a notion of equivalence of C^*-algebras if this is done in two different ways. We also define operations that may be performed on these graphs and then use these operations to determine equivalence of the associated C^*-algebras in certain circumstances. In Chapters 3 and 4 we discuss the Ext functor and show that adding a sink to a graph G determines an element c of $\mathrm{Ext}(C^*(G))$. With this in mind, we construct an isomorphism $\omega : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(A_G - I)$, where A_G is the vertex matrix of G. We also show that the value that ω assigns to c is the class of a vector describing how the sink was added to G. In Chapter 5 we use this isomorphism to strengthen the results of Chapter 2. In particular, if two graphs are formed by adding a sink to G, then we give conditions for their associated C^*-algebras to be equivalent in terms of the vectors describing how the sinks were added.

Contents

Chapter 1

Background and motivation

1.1 History

C^*-algebras are important objects of study in functional analysis. They first appeared in 1947 in a paper of Segal's [89] and were based on the earlier work of Murray and von Neumann on operator algebras in quantum mechanics. The study of C^*-algebras experienced a dramatic and exciting revitalization during the 1970's. Much of this was due to the successes of Brown, Douglas, and Fillmore concerning extensions of C^*-algebras and Elliott's use of K-theory for C^*-algebras. Since that time the study of C^*-algebras has become an extremely active and rapidly expanding subject whose influence has extended to many areas of mathematics. An indication of the significance of the subject is the fact that in recent years Fields Medals have gone to Alain Connes and Vaughan Jones, two researchers who have made major contributions to the area.

Today, the importance of C^*-algebras extends beyond functional analysis and theoretical physics. As algebraic objects that have analytic and topological structure, C^*-algebras form a bridge spanning the gap between algebra and topology. Consequently they have become an indispensable tool for mathematicians working in diverse areas. C^*-algebras have found numerous applications to group representations, dynamical systems, topology, index theory, PDE's, knot theory, and geometry, as well as many other subjects.

It is fair to say that the class of all C^*-algebras is immense. Perhaps this is due to their prolific nature and widespread applicability. In any case, one finds that general C^*-algebras exhibit a variety of behavior and a plethora of characteristics. As a result, those who work in the area find it convenient (or necessary) to consider special classes of C^*-algebras. A variety of such classes has been considered by various authors (e.g. abelian C^*-algebras, AF-algebras, Bunce-Deddens algebras, Cuntz algebras, Toeplitz algebras, irrational rotation algebras, group C^*-algebras, and various crossed products). These classes are important for many reasons: they provide (counter)examples to test hypotheses and conjectures, they allow intermediate stages for proving general results, and they suggest concepts that are often important in studying more general C^*-algebras. The influence of these particular C^*-algebras on the general theory has been enormous. In fact, much of the development of operator algebras in the last twenty years has been based on a careful study of these special classes.

One important and very natural class of examples comes from considering C^*-algebras generated by partial isometries. Due to the theorem of Gelfand and Naimark, any C^*-algebra is isomorphic to a norm closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Partial isometries are very basic examples of operators — they take one subspace (called the initial space) and map it isometrically onto another subspace (called the final space) and are defined to be zero on the orthogonal complement of the initial space. Thus it is natural to take partial isometries and use them as the building blocks for a C^*-algebra. Roughly speaking, this amounts to the following: one begins with a Hilbert space $\mathcal{H}$, chooses some subspaces, and defines partial isometries by selecting an initial space and final space of the same dimension for each. Then one simply considers a C^*-algebra generated by these partial isometries (this can be done, for example, by looking at the subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the partial isometries).

There are a variety of ways to create C^*-algebras by the method just described. In practice, one finds it convenient to choose the final spaces of the partial isometries to be orthogonal, and in many of the examples we consider we shall do exactly that. One also finds it useful to have an object (e.g. a matrix, a graph, etc.) that summarizes how the initial and final spaces of the various partial isometries are related.

Perhaps the best-known example of C^*-algebras created in this way are the Cuntz-Krieger algebras, which are generated by partial isometries whose relations are determined by a finite square matrix with entries in $\{0, 1\}$. Cuntz-Krieger algebras have been generalized in a bewildering number of ways. In the following section we shall describe a few of these generalizations and discuss how they are related to each other.

1.2 C^*-algebras generated by partial isometries and projections

In 1977 J. Cuntz introduced a class of C^*-algebras that became known as Cuntz algebras [12]. For an integer $n \geq 2$ the Cuntz algebra $\mathcal{O}_n$ is defined to be the C^*-algebra generated by isometries $s_1, s_2, \ldots, s_n$ such that

$$\sum_{i=1}^{n} s_i s_i^* = I.$$

It turns out that this C^*-algebra is unique up to isomorphism, and therefore $\mathcal{O}_n$ is well-defined. The Cuntz algebras were important historically because they were the first examples of C^*-algebras whose K-theory has torsion. In fact, Cuntz showed that $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and thus classified the Cuntz algebras by their K-theory [13]. Moreover, the Cuntz algebras have many important properties such as being unital, simple, purely infinite, and nuclear. As such, they are important building blocks in C^*-algebra theory and are also useful for testing and developing hypotheses for more general C^*-algebras.

In 1980 Cuntz and Krieger considered generalized versions of the Cuntz algebras [15]. Rather than having a C^*-algebra for each positive integer n, these C^*-algebras were instead associated to certain square matrices with entries in $\{0,1\}$. If A is an $n \times n$ matrix with entries in $\{0,1\}$, then the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined to be the C^*-algebra generated by partial isometries $s_1, s_2, \ldots, s_n$ with orthogonal ranges that satisfy

$$s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^* \quad \text{for } 1 \leq i \leq n.$$

In order for the isomorphism class of this C^*-algebra to be unique, Cuntz and Krieger required that the matrix A satisfy a nondegeneracy condition called Condition (I). Among other things, Condition (I) implies that A has no zero rows or columns. It turns out that the $n \times n$ matrix A consisting entirely of 1's satisfies Condition (I), and one can see that in this case $\mathcal{O}_A \cong \mathcal{O}_n$. Thus the class of Cuntz-Krieger algebras contains the Cuntz algebras. A study of the Cuntz-Krieger algebras was made in the seminal paper [15] where it was shown that they arise naturally in the study of topological Markov chains. It was also shown that there are important parallels between these C^*-algebras and certain kinds of dynamical systems (e.g. shifts of finite type).

In 1982 Watatani noticed that one could view Cuntz-Krieger algebras as C^*-algebras associated to finite directed graphs [100]. If A is an $n \times n$ matrix with entries in $\{0,1\}$, then the corresponding graph is formed by taking n vertices and drawing an edge from the i^{th} vertex to the j^{th} vertex whenever $A(i,j) = 1$. The fact that A has no zero rows and no zero columns implies that this directed graph has no sinks and no sources. This approach of viewing Cuntz-Krieger algebras as C^*-algebras associated to graphs had the advantage that it was more visual. Instead of working with a matrix, one could now work with a directed graph, and in fact many results about the Cuntz-Krieger algebras took nicer forms in this context.

Although Watatani published some papers using this graph approach [32, 100], his work went largely unnoticed. It was not until 1997 that Kumjian, Pask, Raeburn, and Renault rediscovered C^*-algebras associated to directed graphs. Motivated by their appearance in the duality of compact groups [59], they considered analogues of the Cuntz-Krieger algebras for finite graphs and certain infinite graphs, all of which were allowed to contain sinks. Their original approach involved groupoid techniques [55] that were applied to the path groupoid determined by the graph. Due to technical requirements of the path groupoid, it was necessary for the graphs in question to have no sinks. Soon after, however, they were able to find an approach that avoided the complicated groupoid machinery [54] and could be applied to graphs with sinks. In addition, modern treatments [4] have shown that many of the current results for these C^*-algebras may be obtained by elementary methods that avoid groupoids entirely.

If G is a graph, let G^0 denote its vertices and G^1 denote its edges. Also let $r, s : G^1 \to G^0$ denote the maps identifying the range and source of each edge. If G is a row-finite graph (i.e. all vertices are the source of at most finitely many edges), then we define $C^*(G)$ to be the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in G^0\}$ and partial isometries $\{s_e : e \in G^1\}$ with mutually orthogonal ranges that satisfy

1. $s_e^* s_e = p_{r(e)}$ for all $e \in G^1$

2. $p_v = \displaystyle\sum_{\{e \in G^1 : s(e) = v\}} s_e s_e^*$ whenever v is not a sink.

The condition that G is a row-finite graph was imposed in order to ensure that the sum in Condition 2 is finite. Also note that if v is a sink, then Condition 2 imposes no relation on p_v.

When we say that the projections and partial isometries are "universal", we mean that if A is any C^*-algebra containing projections $\{q_v : v \in G^0\}$ and partial isometries $\{s_e : e \in G^1\}$ that satisfy Conditions 1 and 2, then there exists a homomorphism $\phi : C^*(G) \to A$ such that $\phi(p_v) = q_v$ for all $v \in G^0$ and $\phi(s_e) = s_e$ for all $e \in G^1$. One can easily see that this universality implies that $C^*(G)$ is unique up to isomorphism. Thus by requiring the generators to be universal, one avoids the need for an analogue of Condition (I).

It turns out that the Cuntz-Krieger algebras all arise as C^*-algebras of certain finite graphs with no sinks or sources. (Actually, they are precisely those graphs whose vertex matrix satisfies Condition (I).) Furthermore, the C^*-algebra of a graph with 1 vertex and n edges is the Cuntz algebra $\mathcal{O}_n$.

By allowing certain infinite graphs as well as graphs with sinks and sources, these graph algebras included many C^*-algebras that were not Cuntz-Krieger algebras. However, many people were still unsatisfied with the condition of row-finiteness and longed for a theory of C^*-algebras for arbitrary graphs. This desire was further fueled by the fact that in his original paper [12] Cuntz defined a C^*-algebra $\mathcal{O}_\infty$, which is the universal C^*-algebra generated by a countable sequence of isometries with mutually orthogonal ranges. The C^*-algebra $\mathcal{O}_\infty$ was not included in the class of C^*-algebras of row-finite graphs, and in analogy with the Cuntz algebras it seemed as though $\mathcal{O}_\infty$ should be the C^*-algebra of a graph with one vertex and a countably infinite number of edges. Despite many people's desire to extend the definition of graph algebras to arbitrary graphs, it was unclear exactly how this was to be done. If a vertex is the source of infinitely many edges, then Condition 2 would seem to involve an infinite sum, and it is not clear what this should mean. Because the $s_e s_e^*$'s are orthogonal projections, their sum certainly cannot converge in norm. Additionally, other notions of convergence seemed inappropriate — particularly since one wished to view $C^*(G)$ abstractly and not as represented in a particular way as operators on Hilbert space. Thus the appropriate definition was elusive and it was to be a few years before it could be correctly formulated and a theory of C^*-algebras of arbitrary graphs could be realized.

In the meantime, Exel and Laca took another approach to generalizing the Cuntz-Krieger algebras. In 1999 they published a paper [27] extending the definition of $\mathcal{O}_A$ to infinite matrices. Motivated by the analogy of the Cuntz algebra, it was believed that $\mathcal{O}_\infty$ should correspond to the infinite matrix A consisting entirely of 1's. In the words of Exel and Laca: "In truth $\mathcal{O}_\infty$ is but a beacon, signaling towards a hitherto elusive theory of Cuntz-Krieger algebras for genuinely infinite matrices." If I is any countable (or finite) set and $A = \{A(i,j)_{i,j \in I}\}$ is a $\{0,1\}$-valued matrix over I with no identically zero rows, then the Exel-Laca algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ such that

1. $s_i^* s_i s_j^* s_j = s_j^* s_j s_i^* s_i$ for all $i, j \in I$
2. $s_i s_i^* s_j s_j^* = 0$ when $i \neq j$
3. $s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$ for all $i, j \in I$
4. $\displaystyle\prod_{x \in X} s_x^* s_x \prod_{y \in Y} (1 - s_y^* s_y) = \sum_{j \in I} A(X, Y, j) s_j s_j^*$

whenever X and Y are finite subsets of I such that the function

$$j \in I \mapsto A(X, Y, j) := \prod_{x \in X} A(x, j) \prod_{y \in Y} (1 - A(y, j))$$

is finitely supported.

These Exel-Laca algebras include the C^*-algebras of row-finite graphs without sinks and sources as well as $\mathcal{O}_\infty$.

In 2000 Fowler, Laca, and Raeburn were finally able to extend the definition of graph algebras to arbitrary directed graphs [29]. If G is a graph, let G^0 denote its vertices, G^1 denote its edges, and $r, s : G^1 \to G^0$ denote the maps identifying the range and source of each edge. Then $C^*(G)$ is defined to be the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in G^0\}$ and partial isometries $\{s_e : e \in G^1\}$ with mutually orthogonal ranges that satisfy

1. $s_e^* s_e = p_{r(e)}$ for all $e \in G^1$
2. $p_v = \sum_{\{e \in G^1 : s(e) = v\}} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.
3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in G^1$.

Note that if v is not the source of infinitely many edges, then the relation in Condition 2 implies the relation in Condition 3. Thus, what was needed to extend the definition to arbitrary graphs was to merely impose Condition 3 at vertices emitting infinitely many edges. As with sinks, no relation is imposed by Condition 2 on the vertices that emit infinitely many edges. So in some sense the right condition for extending the definition to arbitrary graphs is almost no condition at all. These graph algebras now included the Cuntz algebra $\mathcal{O}_\infty$, and as expected it arises as the C^*-algebra of the graph with one vertex and infinitely many edges.

Since the Exel-Laca algebras and the C^*-algebras of graphs are both generalizations of the Cuntz-Krieger algebras that are large enough to contain $\mathcal{O}_\infty$, it is natural to wonder how they are related. If G is a graph that has no sinks and no sources, then $C^*(G)$ is an Exel-Laca algebra. In fact, $C^*(G) \cong \mathcal{O}_A$ where A is the edge matrix of G; i.e. the $\{0,1\}$-matrix indexed by the edges of G with $A(e,f) = 1$ if and only if $r(e) = s(f)$. However, there are graphs with sinks and sources whose associated C^*-algebras are not Exel-Laca algebras. In addition, there exist Exel-Laca algebras that are not isomorphic to any graph algebra. It was shown in [78] that if

$$A = \begin{pmatrix} \begin{smallmatrix} 1&0&1&1&1&1 \\ 0&1&1&1&1&1 \\ 1&0&1&0&0&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&0&1 \end{smallmatrix} & \cdots \\ \vdots & \ddots \end{pmatrix}$$

then the Exel-Laca algebra $\mathcal{O}_A$ is not a graph algebra. Thus the Exel-Laca algebras and the graph algebras are incomparable classes of C^*-algebras.

In many ways it is unfortunate that not all Exel-Laca algebras are graph algebras. The defining relations for C^*-algebras of graphs are better motivated and easier to work with than the defining relations for Exel-Laca algebras — in particular, Condition 4 of Exel-Laca algebras is difficult to get a handle on. Also, one finds in practice that graphs are often easier to deal with than matrices and many results take nicer forms when formulated in terms of graphs. Finally, the graph provides a nice tool for visualization which is often absent from the matrix approach. However, despite all the advantages of graphs, there do exist Exel-Laca algebras that are not graph algebras; and if one wishes to study all of these C^*-algebras one cannot abandon the matrix approach entirely. An attempt to deal with this problem was made in [97, 98] where a generalized notion of a graph, called an ultragraph, was defined and it was described how to associate a C^*-algebra to it. It was shown that ultragraph algebras contain both the Exel-Laca algebras and the graph algebras as well as other C^*-algebras not in either of these classes. Although they are more complicated than graph algebras, it was shown in [97, 98] that sometimes the techniques of graph algebras could be modified to prove results about ultragraph algebras.

We summarize the relationships between the various classes of C^*-algebras that we have described:

$$\begin{array}{ccccccccc} \mathrm{CK} & \subset & \mathrm{F}' & \subset & \mathrm{RF}' & \subset & \mathrm{G}' & \subset & \mathrm{EL} = \widetilde{\mathrm{UG}} \\ & & \cap & & \cap & & \cap & & \cap \\ & & \mathrm{F} & \subset & \mathrm{RF} & \subset & \mathrm{G} & \subset & \mathrm{UG} \end{array} \tag{1.1}$$

CK = Cuntz-Krieger algebras $\mathcal{O}_A$ with A satisfying Condition (I)

F′ = C^*-algebras of finite graphs with no sinks or sources

RF′ = C^*-algebras of row-finite graphs with no sinks or sources

= Exel-Laca algebras of row-finite matrices with no zero rows or columns

G′ = C^*-algebras of graphs with no sinks or sources

EL = Exel-Laca algebras

F = C^*-algebras of finite graphs

RF = C^*-algebras of row-finite graphs

G = C^*-algebras of graphs

$\widetilde{\mathrm{UG}}$ = C^*-algebras of ultragraphs with no sinks and in which every vertex emits finitely many edges.

UG = C^*-algebras of ultragraphs

1.3 The role of graph algebras in the theory of C^*-algebras

Graph algebras have proven to be important in the study of C^*-algebras for many reasons. To begin with, they include a fairly wide class of C^*-algebras. In addition to generalizing the Cuntz-Krieger algebras, graph algebras also include many other interesting C^*-algebras. Graph algebras include the compact operators, the Toeplitz algebras, and $C(\mathbb{T})$. In addition, Drinen has shown that up to Morita equivalence all AF-algebras arise as graph algebras [21], and recently Szymański has shown that the continuous functions on quantum spheres, quantum real projective spaces, and quantum complex projective spaces are all graph algebras [36]. Furthermore, Szymański has shown that for any pair of groups (K_0, K_1) with K_1 free, there is a graph algebra whose K-theory is equal to K_0 and K_1. This implies that the class of graph algebras is large, at least as far as K-theory is concerned. In addition, if one considers the C^*-algebras classified by the Kirchberg-Phillips program (i.e. the purely infinite, simple, separable, nuclear C^*-algebras to which the Universal Coefficients Theorem applies), then the work of Szymański in [93] implies that all of these algebras with free K_1-group are Morita equivalent to graph algebras.

Despite the fact that graph algebras include such a wide class of C^*-algebras, their basic structure is fairly well understood and their invariants are readily computable. When working with graph algebras, the graph provides a convenient presentation in terms of generators and relations. Furthermore, the graph not only determines the defining relations for the generators of the C^*-algebra, but also many important properties of the C^*-algebra may be translated into graph properties. Thus the graph provides a tool for visualizing many aspects of the associated C^*-algebra.

In addition, graph algebras are useful examples for much of the work that is currently being done in C^*-algebra theory. We have already mentioned that they have had applications in the duality of compact groups, but many other theories such as Cuntz-Pimsner bimodules, C^*-algebras associated to groupoids, partial actions, and inverse semigroups have benefitted nontrivially from graph algebras. In addition, graph algebras provide useful examples to test the hypotheses of these abstract theories as well as show that they have applications to a wide class of C^*-algebras.

1.4 Topics addressed in this thesis

In this thesis we consider C^*-algebras of row-finite graphs and examine the effect that adding a sink to the graph has on the associated C^*-algebra. In particular, if G is a row-finite graph and E_1 and E_2 are formed by adding a sink to G in two different ways, then we ask: When will $C^*(E_1)$ and $C^*(E_2)$ be "the same"? Understanding the effect that sinks have on the associated C^*-algebra is an important question. Because initial treatments of graph algebras often examined only graphs without sinks, these questions have not been addressed in earlier work. However, now that sinks are allowed in the graphs studied, and because the presence of sinks produces graph algebras that are not Cuntz-Krieger algebras, it is natural to wonder what effect they have on the structure of the C^*-algebra. Furthermore, it was shown in [78] that the C^*-algebra of an infinite graph may be approximated by the C^*-algebras of finite subgraphs containing sinks. As a result, interest has been sparked in the C^*-algebras of graphs with sinks and it has become important to understand them.

In Chapter 2 we examine some constructions that may be performed on the graphs formed by adding a sink to G. In particular, if E_1 and E_2 are formed by adding a sink to a fixed graph G, then we shall see that under certain circumstances it is possible to perform these operations on E_1 and E_2 to produce a common graph F whose C^*-

algebra can be embedded onto full corners of $C^*(E_1)$ and $C^*(E_2)$ in a particular way. This of course implies that $C^*(E_1)$ and $C^*(E_2)$ are Morita equivalent. Many of these results will take the nicest form for finite graphs. In addition, we emphasize that our methods are constructive, and in many cases one can write down the graph F.

In later chapters we extend these results to row-finite (but possibly infinite) graphs. As we shall see, adding a sink to a graph corresponds to creating an extension of $C^*(G)$ by the compact operators $\mathcal{K}$. This leads us naturally into a consideration of $\operatorname{Ext}(C^*(G))$. In Chapter 3 we review the basics of Ext. We also provide a nonstandard description of Ext due to Cuntz and Krieger and prove that it is equivalent to the usual description.

In Chapter 4 we compute Ext for C^*-algebras associated to row-finite graphs in which every loop has an exit. Specifically, we show that $\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(A_G - I)$, where A_G is the vertex matrix of G and is viewed as an endomorphism on $\prod_{G^0} \mathbb{Z}$. Furthermore, if E is a graph formed by adding a sink to G, then we show that the value that the isomorphism assigns to the extension associated to E is the class of a vector in $\operatorname{coker}(A_G - I)$ that describes how the sink was added.

In Chapter 5 we return to the question of determining when $C^*(E_1)$ and $C^*(E_2)$ are the same. Using the isomorphism and the Ext theory developed in Chapter 4 we generalize and strengthen many of the results from Chapter 2. In particular, we show that the graph F described in Chapter 2 can be chosen to be either E_1 or E_2, and thus characterize when one of the $C^*(E_i)$'s may be embedded onto a full corner of the other in a particular way.

A detailed survey of the current work being done in graph algebras is contained in Appendix A. This Appendix was written to serve as a resource for future students who wish to learn about graph algebras. Consequently an effort was made to explain what is currently known about graph algebras and to interpret these results in the context of C^*-algebra theory.

Much of the work in this thesis has been rewritten in papers. The results of Chapter 2 were obtained with D. P. Williams and I. Raeburn and appear in [79], the results of Chapter 4 appear in [95], and the results of Chapter 5 appear in [96].

Chapter 2

Constructions for graphs with sinks

In this chapter we begin our examination of C^*-algebras associated to graphs containing sinks. We shall assume that the reader is comfortable with the basic definitions, terminology, and the notation used for graphs and graph algebras. For the reader unfamiliar with these topics, all of the needed background material may be found in Appendix A. The minimal background needed to read this chapter may be found in Section A.1 and Section A.2.

The results of [4] and [78] show that the presence of sinks can have substantial effects on the structure of a graph algebra, depending on how the sink is attached to the rest of the graph. In order to motivate the approach taken in this thesis, suppose that E is a row-finite graph with a sink v. The set $\{v\}$ is hereditary, and therefore gives rise to an ideal I_v in the C^*-algebra $C^*(E)$ of E. According to general theory, the quotient $C^*(E)/I_v$ can be identified with the graph algebra $C^*(G)$ of the graph G obtained, loosely speaking, by deleting v and all edges which head only into v (see [4, Theorem 4.1]). Therefore, in analyzing the effect that the sink v has on $C^*(E)$, it is important to understand how a sink can be added to G and what kind of C^*-algebras may be produced in this way. Furthermore, one would like to know when adding a sink to G in two different ways produces graphs with similar C^*-algebras.

In light of these observations, we consider primarily graphs E formed by adding a single sink to a fixed graph G, and we call such a graph a *1-sink extension* of G (see Definition 2.1.1). In this chapter and the following we prove some classification theorems describing conditions under which two 1-sink extensions will have C^*-algebras that are Morita equivalent in a particular way. In this way we describe the effect that adding sinks to a graph has on the associated C^*-algebra.

The results in [78] suggest that the appropriate invariant for a 1-sink extension should be the *Wojciech vector* of the extension, which is the element ω_E of $\prod_{G^0} \mathbb{Z}$ whose w^{th} entry is the number of paths in $E^1 \setminus G^1$ from w to the sink. We shall find that the Wojciech vector is an important ingredient in our description, but that it will be equally important to consider the primitive ideal spaces for $C^*(E)$ (or equivalently, the set of maximal tails in E). It turns out that for the special type of Morita equivalence which arises in our study, it will be necessary for the primitive ideal spaces to be homeomorphic in a particular way. In many cases this will be accomplished by requiring our 1-sink extensions to be *essential*; that is, every vertex in G can reach the added sink. More generally, however, we will need to consider the set of maximal tails which can reach the sink.

Many of the results stated in this chapter take the nicest form for finite graphs. In addition, although many of the results stated here will be extended and strengthened in later chapters, we emphasize that these initial results have the advantage that they are constructive in nature. We describe operations which may be performed on the 1-sink extensions to determine whether they are equivalent. For the reader's convenience we now state a special case of our results for the finite graphs which give simple Cuntz-Krieger algebras. We denote by A_G the vertex matrix of a graph G, in which $A_G(w_1, w_2)$ is the number of edges in G from w_1 to w_2.

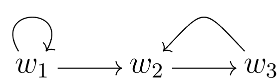
Theorem. *Suppose that E_1 and E_2 are 1-sink extensions of a finite transitive graph G.*

1. *If $\omega_{E_1} - \omega_{E_2} \in \operatorname{im}(A_G - I)$, then there exist a 1-sink extension F of G and embeddings $\phi_i : C^*(F) \to C^*(E_i)$ onto full corners of $C^*(E_i)$ such that the following diagram commutes*

$$\begin{tikzcd}
C^*(F) \arrow[rr, "\phi_i"] \arrow[dr, "\pi_F"'] & & C^*(E_i) \arrow[dl, "\pi_{E_i}"] \\
& C^*(G). &
\end{tikzcd}$$

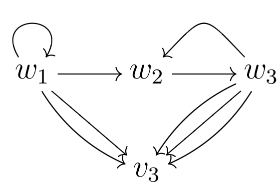
2. *If there exist F and ϕ_i as above, and if $\ker(A_G^t - I) = \{0\}$, then $\omega_{E_1} - \omega_{E_2} \in \operatorname{im}(A_G - I)$.*

While the invariants we are dealing with are K-theoretic in nature, and the proof of Part (2) uses K-theory, we give constructive proofs of Part (1) and of the other main theorems. Thus we can actually find the graph F. For example, if G is given by



and E_1 and E_2 are the 1-sink extensions

then we can take for F the graph



The concrete nature of these constructions is very helpful when we want to apply them to graphs with more than one sink, as we do in §2.4. It also means that our classification is quite different in nature from the K-theoretic classifications of the algebras of finite graphs without sinks [83, 41]. It would be an interesting and possibly very hard problem to combine our theorems with those of [83, 41] to say something about 1-sink extensions of different graphs.

We begin in §2.1 by establishing conventions and notation. We give careful definitions of 1-sink and n-sink extensions, and describe the basic constructions which we use throughout. In §2.2, we consider a class of extensions which we call *essential*; these are the 1-sink extensions E for which the ideal I_v is an essential ideal in $C^*(E)$. For essential 1-sink extensions of row-finite graphs we have a very satisfactory classification (Theorem 2.2.3), which includes Part (1) of the above theorem. We show by example that we cannot completely discard the essentiality, but in §2.3 we extend the analysis to cover non-essential extensions E_1 and E_2 for which the primitive ideal spaces $\operatorname{Prim} C^*(E_1)$ and $\operatorname{Prim} C^*(E_2)$ are appropriately homeomorphic. This extra generality is crucial in §2.4, where we use our earlier results to prove a classification theorem for extensions with n sinks (Theorem 2.4.1). In our last section, we investigate the necessity of our hypothesis on the Wojciech vectors. In particular, Part (2) of the above theorem follows from Corollary 2.5.4.

2.1 Sink extensions and the basic constructions

All graphs in this chapter are row-finite, and unless we say otherwise G will stand for a generic row-finite graph.

Definition 2.1.1. An *n-sink extension* of G is a row-finite graph E which contains G as a subgraph and satisfies:

1. $H := E^0 \setminus G^0$ is finite, contains no sources, and contains exactly n sinks.

2. There are no loops in E whose vertices lie in H.

3. If $e \in E^1 \setminus G^1$, then $r(e) \in H$.

4. If w is a sink in G, then w is a sink in E.

When we say (E, v_i) is an n-sink extension of G, we mean that $v_1, \cdots, v_n$ are the n sinks outside G^0. We consistently write H for $E^0 \setminus G^0$ and S for the set of sinks $\{v_1, \cdots, v_n\}$ lying in H.

If $w \in H$, then there are at most finitely many paths from w to a given sink v_i. If there is one sink v_1 and exactly one path from every $w \in H$ to v_1, we call (E, v_1) a *1-sink tree extension* of G. Equivalently, $(H, s^{-1}(H))$ is a tree.

If we start with a graph E with n sinks, these ideas should apply as follows. Let H be the saturation of the set S of sinks in the sense of [4], and take $G := E \setminus H := (E^0 \setminus H, E^1 \setminus r^{-1}(H))$. Then E satisfies all the above properties with respect to G except possibly (1); if, however, E is finite and has no sources, this is automatic too. So the situation of Definition 2.1.1 is quite general. Property (4) ensures that the saturation of S does not extend into G; it also implies that an m-sink extension of an n-sink extension of G is an $(m+n)$-sink extension of G, which is important for an induction argument in §2.4.

Lemma 2.1.2. *Let (E, v_i) be an n-sink extension of G. Then $H := E^0 \setminus G^0$ is a saturated hereditary subset of E^0. Furthermore, H is the saturation $\overline{S}$ of the set $S := \{v_1, \cdots, v_n\}$.*

Proof. Property (3) of Definition 2.1.1 implies that H is hereditary, and property (4) that H is saturated. Because $\overline{S}$ is the smallest saturated set containing S, it now suffices to prove that $H \subset \overline{S}$. Suppose that $w \notin \overline{S}$. Then either there is a path γ from w to a sink $r(\gamma) \notin S$, or there is an infinite path which begins at w. In the first case, w cannot be in H because $r(\gamma) \notin H$ and H is hereditary. In the second case, w cannot be in H because otherwise we would have an infinite path going round the finite set H, and there would have to be a loop in H. Either way, therefore, $w \notin H$, and we have proved $H \subset \overline{S}$. □

Corollary 2.1.3. *Suppose that (E, v_i) is an n-sink extension of G, and I_S is the ideal in $C^*(E) = C^*(s_e, p_v)$ generated by the projections p_{v_i} associated to the sinks $v_i \in S$. Then there is a surjection π_E of $C^*(E)$ onto $C^*(G) = C^*(t_f, q_w)$ such that $\pi_E(s_e) = t_e$ for $e \in G^1$ and $\pi_E(p_v) = q_v$ for $v \in G^0$, and $\ker \pi_E = I_S$.*

Proof. From Lemma 2.1.2 and [4, Lemma 4.3], we see that $I_S = I_H$, and the result follows from [4, Theorem 4.1]. □

Definition 2.1.4. An n-sink extension (E, v_i) of G is *simple* if $E^0 \setminus G^0 = \{v_i, \cdots, v_n\}$.

We want to associate to each n-sink extension (E, v_i) a simple extension by collapsing paths which end at one of the v_i. For the precise definition, we need some notation. An edge e with $r(e) \in H$ and $s(e) \in G^0$ is called a *boundary edge*; the sources of these edges are called *boundary vertices*. We write B_E^1 and B_E^0 for the sets of boundary edges and vertices. If $v \in G^0$ and $1 \leq i \leq n$, we denote by $Z(v, v_i)$ the set of paths α from v to v_i which leave G immediately in the sense that $r(\alpha_1) \in H$. The *Wojciech vector* of the sink v_i is the element $\omega_{(E;v_i)}$ of $\prod_{G^0} \mathbb{N}$ given by

$$\omega_{(E;v_i)}(v) := \#Z(v, v_i) \text{ for } v \in G^0;$$

notice that $\omega_{(E;v_i)}(v) = 0$ unless v is a boundary vertex. If E has just one sink, we denote its only Wojciech vector by ω_E.

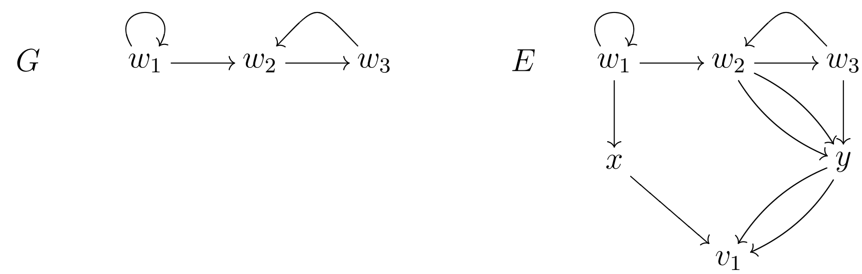
The *simplification* of (E, v_i) is the graph SE with $(SE)^0 := G^0 \cup \{v_1, \cdots, v_n\}$,

$$(SE)^1 := G^1 \cup \{e^{(w,\alpha)} : w \in B_E^0 \text{ and } \alpha \in Z(w, v_i) \text{ for some } i\},$$

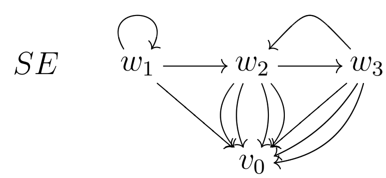
$$s|_{G^1} = s_E, \quad s(e^{(w,\alpha)}) = w, \quad r|_{G^1} = r_E, \quad \text{and} \quad r(e^{(w,\alpha)}) = r(\alpha).$$

The simplification of (E, v_i) is a simple n-sink extension of G with the same Wojciech vectors as E.

Example 2.1.5. Suppose we have the following graph G and the 1-sink extension E.



Then the simplification of E is the following:



We now describe how the C^*-algebra of an n-sink extension is related to the C^*-algebra of its simplification.

Proposition 2.1.6. *Let (E, v_i) be an n-sink extension of G, and let $\{s_e, p_v\}$, $\{t_f, q_w\}$ denote the canonical Cuntz-Krieger families in $C^*(SE)$ and $C^*(E)$. Then there is an embedding ϕ^{SE} of $C^*(SE)$ onto the full corner in $C^*(E)$ determined by the projection $\sum_{i=1}^n q_{v_i} + \sum\{q_w : w \in G^0\}$, which satisfies $\phi^{SE}(p_v) = q_v$ for all $v \in G^0 \cup \{v_i\}$, and* *for which we have a commutative diagram involving the maps π_E of Corollary 2.1.3:*

$$\begin{tikzcd} C^*(SE) \arrow[rr, "\phi^{SE}"] \arrow[dr, "\pi_{SE}"'] & & C^*(E) \arrow[dl, "\pi_E"] \\ & C^*(G) & \end{tikzcd}.$$

Proof. The elements

$$P_v := q_v \quad \text{and} \quad S_e := \begin{cases} t_e & \text{if } e \in G^1 \\ t_\alpha & \text{if } e = e^{(w,\alpha)} \end{cases}$$

form a Cuntz-Krieger (SE)-family in $C^*(E)$, so there is a homomorphism $\phi^{SE} := \pi_{S,P} : C^*(SE) \to C^*(E)$ with $\phi^{SE}(p_v) = P_v$ and $\phi^{SE}(s_e) = S_e$. We trivially have $\phi^{SE}(p_v) = q_v$ for $v \in G^0 \cup S$.

To see that ϕ^{SE} is injective, we use the universal property of $C^*(E)$ to build an action $\beta : \mathbb{T} \to \operatorname{Aut} C^*(E)$ such that

$$\beta_z(q_w) = q_w \quad \text{and} \quad \beta_z(t_f) = \begin{cases} zt_f & \text{if } s(f) \in G^0 \\ t_f & \text{otherwise,} \end{cases}$$

note that ϕ^{SE} converts the gauge action on $C^*(SE)$ to β, and apply the gauge-invariant uniqueness theorem [4, Theorem 2.1].

It follows from [4, Lemma 1.1] that $\sum_{i=1}^{n} q_{v_i} + \sum\{q_w : w \in G^0\}$ converges strictly to a projection $q \in M(C^*(E))$ such that

$$qt_\alpha t_\beta^* = \begin{cases} t_\alpha t_\beta^* & \text{if } s(\alpha) \in G^0 \cup S \\ 0 & \text{otherwise.} \end{cases}$$

Thus $qC^*(E)q$ is spanned by the elements $t_\alpha t_\beta^*$ with $s(\alpha) = s(\beta) \in G^0 \cup S$, and by applying the Cuntz-Krieger relations we may assume $r(\alpha) = r(\beta) \in G^0 \cup S$ also, so that the range of ϕ is the corner $qC^*(E)q$. To see that this corner is full, suppose I is an ideal containing $qC^*(E)q$. Then [4, Lemma 4.2] implies that $K := \{v : q_v \in I\}$ is a saturated hereditary subset of E^0; since K certainly contains $G^0 \cup S$, we deduce that $K = E^0$. But then $I = C^*(E)$ by [4, Theorem 4.1]. Finally, to see that the diagram commutes, we just need to check that π_{SE} and $\pi_E \circ \phi^{SE}$ agree on generators. $\square$

It is convenient to have a name for the situation described in this proposition:

Definition 2.1.7. Suppose (E, v_i) and (F, w_i) are n-sink extensions of G. We say that $C^*(F)$ is $C^*(G)$*-embeddable in* $C^*(E)$ if there is an isomorphism ϕ of $C^*(F) = C^*(s_e, p_v)$ onto a full corner in $C^*(E) = C^*(t_f, q_w)$ such that $\phi(p_{w_i}) = q_{v_i}$ for all i and $\pi_E \circ \phi = \pi_F : C^*(F) \to C^*(G)$. If ϕ is an isomorphism onto $C^*(E)$, we say that $C^*(F)$ is $C^*(G)$*-isomorphic to* $C^*(E)$.

Notice that if $C^*(F)$ is $C^*(G)$-embeddable in $C^*(E)$, then $C^*(F)$ is Morita equivalent to $C^*(E)$ in a way which respects the common quotient $C^*(G)$.

We now describe the basic construction by which we manipulate the Wojciech vectors of graphs.

Definition 2.1.8. Let (E, v_i) be an n-sink extension of G, and let e be a boundary edge such that $s(e)$ is not a source of G. The *outsplitting* of E by e is the graph $E(e)$

defined by

$$E(e)^0 := E^0 \cup \{v'\}; \quad E(e)^1 := (E^1 \setminus \{e\}) \cup \{e'\} \cup \{f' : f \in E^1 \text{ and } r(f) = s(e)\}$$
$$(r,s)|_{E^1\setminus\{e\}} := (r_E, s_E); \quad r(e') := r_E(e),\ s(e') := v'; \quad r(f') := v',\ s(f') := s_E(f).$$

In general, we call $E(e)$ a *boundary outsplitting* of E.

The following example might help fix the ideas:

$$E: \quad \begin{tikzcd} z \arrow[loop left, "h"] \arrow[r, bend right, "f"'] \arrow[d, "e"'] & w \arrow[l, bend right, "g"'] \\ v \end{tikzcd} \qquad\qquad E(e): \quad \begin{tikzcd} z \arrow[loop left, "h"] \arrow[r, bend right, "f"'] \arrow[d, "h'"'] & w \arrow[l, bend right, "g"'] \arrow[dl, bend left, "g'"] \\ v' \arrow[d, "e'"'] \\ v \end{tikzcd}$$

If (E, v_i) is an n-sink extension of G, then every boundary outsplitting $(E(e), v_i)$ is also an n-sink extension of G; if (E, v_0) is a 1-sink tree extension, so is $(E(e), v_0)$. We need to assume that $s(e)$ is not a source of G to ensure that $E(e)$ is an n-sink extension, and we make this assumption implicitly whenever we talk about boundary outsplittings. As the name suggests, boundary outsplittings are special cases of the outsplittings discussed in [58, §2.4].

Proposition 2.1.9. *Suppose $(E(e), v_i)$ is a boundary outsplitting of an n-sink extension (E, v_i) of G. Then $C^*(E(e))$ is $C^*(G)$-isomorphic to $C^*(E)$. If E is a 1-sink tree extension, then the Wojciech vector of $E(e)$ is given in terms of the vertex matrix A_G of G by*

$$\omega_{E(e)} = \omega_E + (A_G - I)\ \delta_{s(e)}. \tag{2.1}$$

Proof. Let $C^*(E) = C^*(t_h, q_w)$. Then

$$P_v := \begin{cases} q_v & \text{if } v \neq s(e) \text{ and } v \neq v' \\ t_e t_e^* & \text{if } v = v' \\ q_{s(e)} - t_e t_e^* & \text{if } v = s(e) \end{cases}$$

$$S_g := \begin{cases} t_e & \text{if } g = e' \\ t_g(q_{s(e)} - t_e t_e^*) & \text{if } g \neq e' \text{ and } r(g) = s(e) \\ t_f t_e t_e^* & \text{if } g = f' \text{ for some } f \in E^1 \text{ with } r(f) = s(e) \\ t_g & \text{otherwise} \end{cases}$$

is a Cuntz-Krieger $E(e)$-family which generates $C^*(E)$. The universal property of $C^*(E(e)) = C^*(s_g, p_v)$ gives a homomorphism $\phi = \pi_{S,P} : C^*(E(e)) \to C^*(E)$ such that $\phi(s_g) = S_g$ and $\phi(p_v) = P_v$, which is an isomorphism by the gauge-invariant uniqueness theorem [4, Theorem 2.1]. It is easy to check on generators that ϕ is a $C^*(G)$-isomorphism.

When H is a tree with one sink v_0, there is precisely one path γ in E from $r(e)$ to v_0, and hence all the new paths from a vertex v to v_0 have the form $f'\gamma$. Thus if $v \neq s(e)$,

$$\begin{aligned} \omega_{E(e)}(v) &= \omega_E(v) + \#\{f' \in E(e)^1 : s(f') = v \text{ and } f' \notin E^1\} \\ &= \omega_E(v) + \#\{f \in G^0 : s(f) = v \text{ and } r(f) = s(e)\} \\ &= \omega_E(v) + A_G(v, s(e)). \end{aligned}$$

On the other hand, if $v = s(e)$, then

$$\begin{aligned}
\omega_{E(e)}(s(e)) &= \omega_E(s(e)) + \#\{f' \in E(e)^1 : s(f') = s(e) \text{ and } f' \notin E^1\} - 1 \\
&= \omega_E(s(e)) + \#\{f \in G^0 : s(f) = s(e) = r(f)\} - 1 \\
&= \omega_E(v) + A_G(s(e), s(e)) - 1.
\end{aligned}$$

Together these calculations give (2.1). □

Suppose that $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ is a path in G and there is a boundary edge e with $s(e_1) = r(\alpha)$. Then $E(e)$ will have a boundary edge α'_n at $r(\alpha_{n-1})$, and therefore we can outsplit again to get $E(e)(\alpha'_n)$. This graph has a boundary edge α'_{n-1} at $r(\alpha_{n-2})$, and we can outsplit again. Continuing this process gives an extension $E(e, \alpha)$ in which $s(\alpha)$ is a boundary vertex. We shall refer to this process as *performing outsplittings along the path α.* From Proposition 2.1.9 we can calculate the Wojciech vector of $E(e, \alpha)$:

Corollary 2.1.10. *Suppose E is a 1-sink tree extension of G and α is a path in G for which $r(\alpha)$ is a boundary vertex. Then for any boundary edge e with $s(e) = r(\alpha)$, we have*

$$\omega_{E(e,\alpha)} = \omega_E + \sum_{i=1}^{|\alpha|} (A_G - I)\delta_{r(\alpha_i)}.$$

2.2 A classification for essential 1-sink extensions

We now ask to what extent the Wojciech vector determines a 1-sink extension. Suppose that E_1 and E_2 are 1-sink extensions of G. Our main results say, loosely speaking, that if the Wojciech vectors ω_{E_i} determine the same class in $\operatorname{coker}(A_G - I)$, then there will be a simple extension F such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$

and $C^*(E_2)$. However, we shall need some hypotheses on the way the sinks are attached to G; the hypotheses in this section are satisfied if, for example, G is one of the finite transitive graphs for which $C^*(G)$ is a simple Cuntz-Krieger algebra. We begin by describing the class of extensions which we consider in this section.

Recall that if v, w are vertices in G, then $v \geq w$ means there is a path γ with $s(\gamma) = v$ and $r(\gamma) = w$. For $K, L \subset G^0$, $K \geq L$ means that for each $v \in K$ there exists $w \in L$ such that $v \geq w$. If γ is a loop, we write $\gamma \geq L$ when $\{r(\gamma_i)\} \geq L$.

Definition 2.2.1. A 1-sink extension (E, v_0) of a graph G is an *essential extension* if $G^0 \geq v_0$.

We can see immediately that simplifications of essential extensions are essential extensions, and consideration of a few cases shows that boundary outsplittings of essential extensions are essential extensions. To see why we chose the name, recall that an ideal I in a C^*-algebra A is essential if $I \cap J \neq 0$ for all nonzero ideals J in A, or equivalently, if $aI = 0$ implies $a = 0$. Then we have:

Lemma 2.2.2. *Let (E, v_0) be a 1-sink extension of G. Then (E, v_0) is an essential extension of G if and only if the ideal I_{v_0} generated by p_{v_0} is an essential ideal in $C^*(E) = C^*(s_e, p_v)$.*

Proof. Suppose that there exists $w \in G^0$ such that $w \not\geq v_0$. Then since

$$I_{v_0} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_0\}.$$

(see [4, Lemma 4.3]), we have $p_w I_{v_0} = 0$, and I_{v_0} is not essential.

Conversely, suppose that $G^0 \geq v_0$. To show that I_{v_0} is an essential ideal it suffices to prove that if $\pi : C^*(G) \to B(\mathcal{H})$ is a representation with $\ker \pi \cap I_{v_0} = \{0\}$, then π is faithful. So suppose $\ker \pi \cap I_{v_0} = \{0\}$. In particular, we have $\pi(s_{v_0}) \neq 0$. For every $v \in G^0$ there is a path α in E such that $s(\alpha) = v$ and $r(\alpha) = v_0$. Then $\pi(s_\alpha^* s_\alpha) = \pi(p_{v_0}) \neq 0$, and hence $\pi(p_v) \geq \pi(s_\alpha s_\alpha^*) \neq 0$. Since every loop in a 1-sink extension E must lie entirely in G, every loop in G has an exit in E; thus we can apply [4, Theorem 3.1] to deduce that π is faithful, as required. $\square$

We can now state our classification theorem for essential extensions.

Theorem 2.2.3. *Let G be a row-finite graph with no sources, and suppose that (E_1, v_1) and (E_2, v_2) are essential 1-sink extensions of G with finitely many boundary vertices. If there exists $n \in \bigoplus_{G^0} \mathbb{Z}$ such that the Wojciech vectors satisfy $\omega_{E_1} - \omega_{E_2} = (A_G - I)n$, then there is a simple 1-sink extension F of G such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$.*

We begin by observing that, since a full corner in a full corner of a C^*-algebra A is a full corner in A, the composition of two $C^*(G)$-embeddings is another $C^*(G)$-embedding. Thus it suffices by Proposition 2.1.6 to prove the theorem for the simplifications SE_1 and SE_2. However, since we are going to perform boundary outsplittings and these do not preserve simplicity, we assume merely that E_1 and E_2 are 1-sink tree extensions. The following lemma is the key to many of our constructions:

Lemma 2.2.4. *Let (E_1, v_1) and (E_2, v_2) be 1-sink tree extensions of G with finitely many boundary vertices, and suppose that $B^0_{E_1} \geq B^0_{E_2} \geq B^0_{E_1}$. If γ is a loop in G such that $\gamma \geq B^0_{E_1}$, then for any $a \in \mathbb{Z}$ there are 1-sink tree extensions E_1' and E_2' which are formed by performing a finite number of boundary outsplittings to E_1 and E_2, respectively, and for which*

$$\omega_{E_1'} - \omega_{E_2'} = \omega_{E_1} - \omega_{E_2} + a\Big(\sum_{j=1}^{|\gamma|} (A_G - I)\delta_{r(\gamma_j)}\Big).$$

Proof. Since the statement is symmetric in E_1 and E_2, it suffices to prove this for $a > 0$. Choose a path α in G such that $s(\alpha) = r(\gamma)$ and $r(\alpha) \in B^0_{E_1}$. Since $B^0_{E_1}$ is finite, going along paths from $r(\alpha)$ to $B^0_{E_2}$ and then to and fro between $B^0_{E_2}$ and $B^0_{E_1}$ must eventually give a loop μ which visits both $B^0_{E_1}$ and $B^0_{E_2}$, and a path β with $s(\beta) = r(\alpha)$ and $r(\beta) = s(\mu) \in B^0_{E_1}$. Since there are boundary edges $e_1 \in B^1_{E_1}$ and $e_2 \in B^1_{E_2}$ with $s(e_i)$ on μ, we can perform outsplittings along μ to get new tree extensions $E_i(e_i, \mu^i)$, where μ^i is the loop μ relabeled so that it ends at $s(e_i)$. Because μ^1 and μ^2 have the same vertices as μ in a different order, Corollary 2.1.10 gives

$$\omega_{E_i(e_i,\mu^i)} = \omega_{E_i} + \sum_{j=1}^{|\mu|}(A_G - I)\delta_{r(\mu_j)},$$

so we have $\omega_{E_1(e_1,\mu^1)} - \omega_{E_2(e_2,\mu^2)} = \omega_{E_1} - \omega_{E_2}$. Since $r(\beta_{|\beta|}) = s(\mu)$, and in forming both $E_i(e_i, \mu^i)$ we have performed an outsplitting at $s(\mu)$, $s(\beta_{|\beta|})$ is a boundary vertex in both $E_i(e_i, \mu^i)$; say $f_i \in B^1_{E_i}$ has $s(f_i) = s(\beta_{|\beta|})$. Write $\beta = \beta'\beta_{|\beta|}$, let γ^a be the path obtained by going a times around γ, and define

$$E_1' := E_1(e_1, \mu^1)(f_1, \gamma^a\alpha\beta'), \quad E_2' := E_2(e_2, \mu^2)(f_2, \alpha\beta').$$

We now compute the Wojciech vectors using Corollary 2.1.10: for example,

$$\omega_{E_1'} = \omega_{E_1(e_1,\mu^1)} + (A_G - I)\Big(\sum_{j=1}^{|\beta|-1}\delta_{r(\beta_j)} + \sum_{j=1}^{|\alpha|}\delta_{r(\alpha_j)} + \sum_{j=1}^{|\gamma|}a\delta_{r(\gamma_j)}\Big).$$

The formula for $\omega_{E_2'}$ is the same except for the last term, so

$$\begin{aligned}\omega_{E_1'} - \omega_{E_2'} &= \omega_{E_1(e_1,\mu^1)} - \omega_{E_2(e_2,\mu^2)} + \sum_{j=1}^{|\gamma|}a(A_G - I)\delta_{r(\gamma_j)}\\ &= \omega_{E_1} - \omega_{E_2} + \sum_{j=1}^{|\gamma|}a(A_G - I)\delta_{r(\gamma_j)},\end{aligned}$$

as required. □

Proof of Theorem 2.2.3. As we indicated earlier, it suffices to prove the theorem when E_1 and E_2 are tree extensions. It also suffices to prove that we can perform boundary outsplittings on E_1 and E_2 to achieve extensions F_1 and F_2 with the same Wojciech vector; Propositions 2.1.6 and 2.1.9 then imply that we can take for F the common simplification of F_1 and F_2. We can write $n = \sum_{k=1}^{m} a_k \delta_{w_k}$ for some finite set $\{w_1, w_2, \ldots, w_m\} \subset G^0$. We shall prove by induction on m that we can perform the required outsplittings. If $m = 0$, then $\omega_{E_1} = \omega_{E_2}$, and there is nothing to prove. So we suppose that we can perform the outsplittings whenever n has the form $\sum_{k=1}^{m} a_k \delta_{w_k}$, and that $n = \sum_{k=1}^{m+1} a_k \delta_{w_k}$. Let D be the subgraph of G with vertices $D^0 := \{w_1, w_2, \ldots, w_{m+1}\}$ and edges $D^1 := \{e \in G^1 : s(e), r(e) \in D^0\}$. Since D is a finite graph it contains either a sink or a loop.

If D contains a sink, then by relabeling we can assume the sink is w_{m+1}. Since $A_G(w_{m+1}, w_j) = 0$ for all j, we have

$$\omega_{E_1}(w_{m+1}) = \omega_{E_2}(w_{m+1}) - a_{m+1}.$$

Thus either E_1 or E_2 has at least $|a_{m+1}|$ boundary edges leaving w_{m+1}: we may as well assume that $a_{m+1} > 0$, so that $\omega_{E_2}(w_{m+1}) \geq a_{m+1}$. We can then perform a_{m+1} boundary outsplittings on E_2 at w_{m+1} to get a new extension E_2'. From Proposition 2.1.9, we have $\omega_{E_2'} = \omega_{E_2} + a_{m+1}(A_G - I)\delta_{w_{m+1}}$, and therefore

$$\omega_{E_1} = \omega_{E_2'} + (A_G - I)\Big(\sum_{k=1}^{m} a_k \delta_{w_k}\Big).$$

Since E_2' is formed by performing boundary outsplittings to the essential tree extension E_2, it is also an essential tree extension, and the inductive hypothesis implies that we can perform boundary outsplittings on E_1 and E_2' to arrive at extensions with the same Wojciech vector.

If D does not have a sink, it must contain a loop γ. If necessary, we can shrink γ so that its vertices are distinct, and by relabeling, we may assume that w_{m+1} lies on γ. Because the extensions are essential, we have $G^0 \geq B^0_{E_1}$ and $G^0 \geq B^0_{E_2}$, so we can apply Lemma 2.2.4. Thus there are 1-sink tree extensions E_1' and E_2' formed by performing boundary outsplittings to E_1 and E_2, and for which

$$\omega_{E_1'} - \omega_{E_2'} = \omega_{E_1} - \omega_{E_2} - a_{m+1} \sum_{j=1}^{|\gamma|} (A_G - I)\delta_{r(\gamma_j)}.$$

But because $\omega_{E_1} = \omega_{E_2} + (A_G - I)n$ this implies that

$$\omega_{E_1'} = \omega_{E_2'} + (A_G - I)\Big(\sum_{j=1}^{m} b_j \delta_{w_j}\Big),$$

where $b_j = a_j - a_{m+1}$ if w_j lies on γ, and $b_j = a_j$ otherwise. We can now invoke the inductive hypothesis to see that we can perform boundary outsplittings to E_1' and E_2' to arrive at extensions with the same Wojciech vector.

This completes the proof of the inductive step, and the result follows. $\square$

Remark 2.2.5. The graph F in Theorem 2.2.3 has actually been constructed in a very specific way, and it will be important in Section 2.4 that we can keep track of the procedures used. We shall say that one simple extension F has been obtained from another E *by a standard construction* if it is the simplification of a graph obtained by performing a sequence of boundary outsplittings to E. The graph F in Theorem 2.2.3 has been obtained from both SE_1 and SE_2 by a standard construction.

In later chapters we shall see that the hypotheses of Theorem 2.2.3 are much stronger than necessary. In particular, the result will hold when G contains sources.

However the following example shows that when G contains sources the graph F cannot always be obtained by a standard construction.

Example 2.2.6. Let G be the graph

$$\begin{tikzcd} w_1 \arrow[r] & w_2 \arrow[loop, out=150, in=90, looseness=5] \arrow[loop, out=90, in=30, looseness=5] & w_3 \arrow[l] \end{tikzcd}$$

with extensions E_1 and E_2

$$E_1 : \begin{tikzcd} w_1 \arrow[r] \arrow[dr] & w_2 \arrow[loop, out=150, in=90, looseness=5] \arrow[loop, out=90, in=30, looseness=5] \arrow[d] & w_3 \arrow[l] \arrow[dl] \\ & v_1 & \end{tikzcd} \qquad E_2 : \begin{tikzcd} w_1 \arrow[r] \arrow[dr] \arrow[dr, bend right] & w_2 \arrow[loop, out=150, in=90, looseness=5] \arrow[loop, out=90, in=30, looseness=5] \arrow[d] & w_3 \arrow[l] \arrow[dl] \arrow[dl, bend left] \\ & v_1 & \end{tikzcd}$$

Then we see that $\omega_{E_1} = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ and $\omega_{E_2} = \left(\begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}\right)$. Since the only vertex of G that is not a source is w_2, we may only perform boundary outsplittings to E_1 and E_2 at w_3. Now if k boundary outsplittings are performed on E_1, then the resulting extension will have Wojciech vector equal to

$$\omega_{E_1} + k(A_G - I)\delta_{w_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + k \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+k \\ 1+k \\ 1+k \end{pmatrix}.$$

Similarly, if l outsplittings are performed on E_2, then the resulting extension will have Wojciech vector equal to

$$\omega_{E_2} + l(A_G - I)\delta_{w_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + l \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+l \\ 2+l \\ 1+l \end{pmatrix}.$$

Because these vectors are not equal for any values of k and l, we see that E_1 and E_2

cannot be boundary outsplit to extensions with a common reduction.

A more important hypothesis in Theorem 2.2.3 is the essentiality of the 1-sink extensions. Although we shall prove an analogue of Theorem 2.2.3 in which the condition of essentiality is weakened, the next example that it cannot be completely dropped.

Example 2.2.7. Consider the following graph G

$$\begin{tikzcd} w_1 \arrow[loop above] \arrow[loop below] \arrow[r] & w_2 \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

and its extensions E_1 and E_2;

$$E_1 : \begin{tikzcd} w_1 \arrow[loop above] \arrow[loop below] \arrow[r] \arrow[d] & w_2 \arrow[loop above] \arrow[loop below] \\ v_1 & \end{tikzcd} \qquad\qquad E_2 : \begin{tikzcd} w_1 \arrow[loop above] \arrow[loop below] \arrow[r] & w_2 \arrow[loop above] \arrow[loop below] \arrow[d] \\ & v_2 \end{tikzcd}$$

Note that E_2 is essential but E_1 is not. On one hand, we have $A_G = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$, $\omega_{E_1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $\omega_{E_2} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, so

$$\omega_{E_1} - \omega_{E_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (A_G - I) \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

On the other hand, we claim that $C^*(E_1)$ is not Morita equivalent to $C^*(E_2)$, so that they cannot have a common full corner. To see this, recall from [4, Theorem 4.4] that the ideals in $C^*(E_i)$ are in one-to-one correspondence with the saturated hereditary subsets of E_i^0. The saturated hereditary subsets of E_1^0 are $\{v_1\}$, $\{v_1, w_2\}$, $\{v_1, w_1, w_2\}$ and $\{w_2\}$, and those of E_2^0 are $\{v_2\}$, $\{v_2, w_2\}$ and $\{v_2, w_1, w_2\}$. Thus $C^*(E_1)$ has more ideals than $C^*(E_2)$. But if they were Morita equivalent, the Rieffel correspondence would set up a bijection between their ideal spaces.

This example shows that the way the sinks v_i are attached to G can affect how the ideal I_{v_0} lies in the ideal space of $C^*(E)$. In the next section, we give a simple condition on the way v_i are attached which ensures that the primitive ideal spaces of $C^*(E_i)$ are homeomorphic, and show that under this condition there is a good analogue of Theorem 2.2.3. However, there is one situation in which essentiality is not needed: when $C^*(G)$ is an AF-algebra.

Corollary 2.2.8. *Let G be a graph with no sources for which $C^*(G)$ is an AF-algebra, and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If there exists $n \in \bigoplus_{G^0} \mathbb{Z}$ such that $\omega_{E_1} = \omega_{E_2} + (A_G - I)n$, then there is a simple 1-sink extension F of G such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$.*

Proof. We first recall from [54, Theorem 2.4] that $C^*(G)$ is AF if and only if G has no loops. Now we proceed as in the proof of Theorem 2.2.3. Everything goes the same until we come to consider the finite subgraph D associated to the support of the vector n. Since there are no loops in G, D must have a sink, and the argument in the second paragraph of the proof of Theorem 2.2.3 suffices; this does not use essentiality. □

2.3 A classification for non-essential 1-sink extensions

Recall from [4, §6] that a *maximal tail* in a graph E is a nonempty subset of E^0 which is cofinal under $\geq$, is backwards hereditary ($v \geq w$ and $w \in \gamma$ imply $v \in \gamma$), and contains no sinks (for each $w \in \gamma$, there exists $e \in E^1$ with $s(e) = w$ and $r(e) \in \gamma$).

Definition 2.3.1. Let (E, v_0) be a 1-sink extension of G. The *closure* of the sink v_0 is the set

$$\overline{v_0} := \bigcup\{\gamma : \gamma \text{ is a maximal tail in } G \text{ and } \gamma \geq v_0\}.$$

To motivate this definition, we notice first that the extension is essential if and only if $\overline{v_0} = G^0$. More generally (although it is not logically necessary for our results), we explain how this notion of closure is related to the closure of sets in $\operatorname{Prim} C^*(E)$, as described in [4, §6]. For each sink v, let $\lambda_v := \{w \in E^0 : w \geq v\}$, and let

$$\Lambda_E := \{\text{maximal tails in } E\} \cup \{\lambda_v : v \text{ is a sink in } E\}.$$

The set Λ_E has a topology in which the closure of a subset S is $\{\lambda : \lambda \geq \bigcup_{\chi \in S} \chi\}$, and it is proved in [4, Corollary 6.5] that when E satisfies Condition (K) of [55], $\lambda \mapsto I_{E^0 \setminus \lambda}$ is a homeomorphism of Λ_E onto $\operatorname{Prim} C^*(E)$. If (E, v_0) is a 1-sink extension of G, then the only loops in E are those in G, so E satisfies Condition (K) whenever G does. A subset of G^0 is a maximal tail in E if and only if it is a maximal tail in G, and because every sink in G is a sink in E, we deduce that $\Lambda_E = \Lambda_G \cup \{\lambda_{v_0}\}$.

Lemma 2.3.2. *Suppose G satisfies Condition* (K)*, and* (E_1, v_1)*,* (E_2, v_2) *are 1-sink extensions of G. Then $\overline{v_1} = \overline{v_2}$ if and only if there is a homeomorphism h of* $\operatorname{Prim} C^*(E_1)$ *onto* $\operatorname{Prim} C^*(E_2)$ *such that*

$$h(I_{E_1^0 \setminus \lambda}) = I_{E_2^0 \setminus \lambda} \text{ for } \lambda \in \Lambda_G, \text{ and } h(I_{E_1^0 \setminus \lambda_{v_1}}) = I_{E_2^0 \setminus \lambda_{v_2}}. \tag{2.2}$$

Proof. For any 1-sink extension (E, v_0), the map $J \mapsto \pi_E^{-1}(J)$ is a homeomorphism of $\operatorname{Prim} C^*(G)$ onto the closed subset $\{J \in \operatorname{Prim} C^*(E) : J \supset I_{v_0}\}$. If $\lambda \in \Lambda_G \subset \Lambda_E$, then $\pi_E^{-1}(I_{G^0 \setminus \lambda}) = I_{E^0 \setminus \lambda}$, and hence h is always a homeomorphism of the closed set $\{I_{E_1^0 \setminus \lambda} : \lambda \in \Lambda_G\}$ in $\operatorname{Prim} C^*(E_1)$ onto the corresponding subset of $\operatorname{Prim} C^*(E_2)$. So the only issue is whether the closures of the sets $I_{E_1^0\setminus\lambda_{v_1}}$ and $I_{E_2^0\setminus\lambda_{v_2}}$ match up. But

$$\overline{I_{E_i^0\setminus\lambda_{v_i}}} = \{I_{E_i^0\setminus\lambda} : \lambda \geq \lambda_{v_i}\} = \{I_{E_i^0\setminus\lambda} : \lambda \geq v_i\}.$$

Since other sets λ_v associated to sinks are never $\geq v_i$, the ideals on the right-hand side are those associated to the maximal tails lying in $\overline{v_i}$, and the result follows. □

We now return to the problem of proving analogues of Theorem 2.2.3 for non-essential extensions. Notice that the closure is a subset of G^0 rather than E^0: we have defined it this way because we want to compare the closures in different extensions.

Proposition 2.3.3. *Suppose that (E_1, v_1) and (E_2, v_2) are 1-sink extensions of G with finitely many boundary vertices, and suppose that $\overline{v_1} = \overline{v_2} = C$, say. If $\omega_{E_1} - \omega_{E_2}$ has the form $(A_G - I)n$ for some $n \in \bigoplus_C \mathbb{Z}$ (where $\bigoplus_C \mathbb{Z}$ is viewed as a subset of $\bigoplus_{G^0} \mathbb{Z}$), then there there is a simple 1-sink extension F of G such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$.*

We aim to follow the proof of Theorem 2.2.3, so we need to check that the operations used there will not affect the hypotheses in Proposition 2.3.3. It is obvious that the closure is unaffected by simplifications. It is true but not so obvious that it is unaffected by boundary outsplittings:

Lemma 2.3.4. *Suppose (E, v_0) is a 1-sink extension of a graph G, and e is a boundary edge in E. Then the closures of v_0 in E and $E(e)$ are the same.*

Proof. Suppose γ is a maximal tail such that $\gamma \geq v_0$ in $E(e)$ and $z \in \gamma$; we want to prove $z \geq B_E^0$. We know $z \geq w$ for some $w \in B_{E(e)}^0$. If $w \in B_E^0$, there is no problem. If $w \notin B_E^0$, then $w = s(f)$ for some $f \in G^1$ with $r(f) = s(e)$, so $z \geq w \geq s(e) \in B_E^0$.

Now suppose $\gamma \geq v_0$ in E and $z \in \gamma$; we want to prove that $z \geq B_{E(e)}^0$. We know that there is a path α with $s(\alpha) = z$ and $r(\alpha) \in B_E^0$. If $r(\alpha) \neq s(e)$, we have

$z \geq r(\alpha) \in B^0_{E(e)}$. If $r(\alpha) = s(e)$ and $|\alpha| \geq 1$, we have $z \geq r(\alpha_{|\alpha|-1}) \in B^0_{E(e)}$. The one remaining possibility is that $z = s(e)$ and there is no path of length at least 1 from $s(e)$ to $s(e)$. Because γ is a tail, there exists $f \in G^1$ such that $s(f) = s(e)$ and $r(f) \in \gamma$. Now we use $\gamma \geq v_0$ to get a path β with $s(\beta) = r(f)$ and $r(\beta) \in B^0_{E^0} \setminus \{s(e)\}$, and we are back in the first case with $\alpha = f\beta$. □

Proof of Proposition 2.3.3. Since the closures $\overline{v_1}$ and $\overline{v_2}$ are unaffected by simplification and boundary outsplitting, we can run the argument of Theorem 2.2.3. In doing so, we never have to leave the common closure C: by hypothesis, $n = \sum_{k=1}^m a_k w_k$ for some $w_k \in C$, so all the vertices on the subgraph D used in the inductive step lie in C. When D has a sink, the argument goes over verbatim. When D has a loop γ, all the vertices on γ lie in C, and the hypothesis $\overline{v_1} = C = \overline{v_2}$ implies that $\gamma \geq B^0_{E_1} \geq B^0_{E_2} \geq B^0_{E_1}$, so we can still apply Lemma 2.2.4. The rest of the argument carries over. □

The catch in Proposition 2.3.3 is that the vector n is required to have support in the common closure C. For our applications to n-sink extensions in the next section, this is just what we need. However, if we are only interested in 1-sink extensions, this requirement might seem a little unnatural. So it is interesting that we can often remove it:

Lemma 2.3.5. *Suppose that (E_1, v_1) and (E_2, v_2) are 1-sink extensions of G, and suppose that $\overline{v_1} = \overline{v_2} = C$, say. Suppose that 1 is not an eigenvalue of the $(G^0 \setminus C) \times (G^0 \setminus C)$ corner of A_G. Then if $\omega_{E_1} - \omega_{E_2}$ has the form $(A_G - I)n$ for some $n \in \bigoplus_{G^0} \mathbb{Z}$, we have $n \in \bigoplus_C \mathbb{Z}$.*

Proof. Since the maximal tails comprising C are backwards hereditary, there are no paths from $G^0 \setminus C$ to C. Thus A_G decomposes with respect to the decomposition

$G^0 = (G^0 \setminus C) \cup C$ as $A_G = \left(\begin{smallmatrix} B & 0 \\ C & D \end{smallmatrix}\right)$, and $A_G - I = \left(\begin{smallmatrix} B-I & 0 \\ C & D-I \end{smallmatrix}\right)$. Writing n as $\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)$ and noting that $\omega_{E_1} - \omega_{E_2}$ has support in C shows that $(B - I)k = 0$, which by the hypothesis on A_G implies $k = 0$. But this says exactly what we want. $\square$

2.4 A classification for n-sink extensions

We say that an n-sink extension is *essential* if $G^0 \geq v_i$ for $1 \leq i \leq n$.

Theorem 2.4.1. *let (E, v_i) and (F, w_i) be essential n-sink extensions of G with finitely many boundary vertices. Suppose that the Wojciech vectors satisfy*

$$\omega_{(E;v_i)} - \omega_{(F;w_i)} \in (A_G - I)\Big(\bigoplus_{G^0}\mathbb{Z}\Big) \text{ for } 1 \leq i \leq n. \tag{2.3}$$

Then there is a simple n-sink extension D of G such that $C^*(D)$ is $C^*(G)$-embeddable in both $C^*(E)$ and $C^*(F)$.

We shall prove this theorem by induction on n. At a key point we need to convert $(n-1)$-sink extensions to n-sink extensions. If $m \in \prod_{G^0} \mathbb{N}$ and (E, v_i) is an $(n-1)$-sink extension, we denote by $(E * m, v_i)$ the n-sink extension of G obtained by adding an extra vertex v_n and $m(w)$ edges from each vertex $w \in G^0$ to v_n. Note that $E * m$ has one new Wojciech vector $\omega_{(E*m;v_n)} = m$, and the other Wojciech vectors are unchanged. If E is a simple extension, then so is $E * m$. Conversely, if (F, w_i) is a simple n-sink extension, then $F \setminus w_n := (F^0 \setminus \{w_n\}, F^1 \setminus r^{-1}(w_n))$ is a simple $(n-1)$-sink extension for which $(F \setminus w_n) * \omega_{(F;w_n)}$ can be naturally identified with F.

We need to know how the operation $E \mapsto E * m$ interacts with our other constructions:

Lemma 2.4.2. *If e is a boundary edge for E, then e is a boundary edge for $E * m$,* *and the boundary outsplittings satisfy $E(e) * m = (E * m)(e)$. The simplification construction $E \mapsto SE$ satisfies $S(E * m) = (SE) * m$.*

Proof. The only edges which are affected in forming $E(e)$ are e and the edges f with $r(f) = s(e)$. Since none of the new edges in $E * m$ have range in E, they are not affected by the outsplitting. Simplifying collapses paths which end at one of the sinks v_i, and forming $E * m$ adds only paths of length 1 ending at v_n, so there is nothing extra to collapse in simplifying $E * m$. $\square$

Proof of Theorem 2.4.1. As in the 1-sink case, it suffices by Proposition 2.1.6 to prove the result when E and F are simple. So we assume this. Our proof is by induction on n, but we have to be careful to get the right inductive hypothesis. So we shall prove that by performing n standard constructions on both E and F, we can arrive at simple n-sink extensions of G with all their Wojciech vectors equal; these graphs are then isomorphic, and we can take D to be either of them. Theorem 2.2.3 says that this is true for $n = 1$ (see Remark 2.2.5).

So we suppose that our inductive hypothesis holds for all simple $(n-1)$-sink extensions satisfying the hypotheses of Theorem 2.4.2. Then $E \setminus v_n$ and $F \setminus w_n$ are simple $(n-1)$-sink extensions of G with Wojciech vectors $\omega_{(E \setminus v_n; v_i)} = \omega_{(E; v_i)}$ and $\omega_{(F \setminus w_n; w_i)} = \omega_{(F; w_i)}$ for $i \leq n-1$. So the Wojciech vectors of $E \setminus v_n$ and $F \setminus w_n$ satisfy the hypothesis (2.3). Since $G^0 \geq v_i$ in E, and we have not deleted any edges except those ending at v_n and w_n, we still have $G^0 \geq v_i$ in $E \setminus v_n$ for $i \leq n-1$, and similarly $G^0 \geq w_i$ in $F \setminus w_n$. By the inductive hypothesis, therefore, we can perform $(n-1)$ standard constructions on each of E and F to arrive at the same simple $(n-1)$-sink extension (D, u_i) of G.

By Lemma 2.4.2, $D * \omega_{(E; v_n)}$ and $D * \omega_{(F; w_n)}$ are obtained from $E = (E \setminus v_n) * \omega_{(E; v_n)}$ and $F = (F \setminus w_n) * \omega_{(F; w_n)}$ by $(n-1)$ standard constructions. We now view $(D^E, v_n) :=$

$D * \omega_{(E;v_n)}$ and $(D^F, w_n) := D * \omega_{(F;w_n)}$ as two simple 1-sink extensions of the graph D. Since the standard constructions have not affected the path structure of G inside D, and we assumed $G^0 \geq v_n$ in E, we still have $G^0 \geq v_n$ in D^E, and similarly $G^0 \geq w_n$ in D^F. Because any sink in G has to be a sink in E, the hypothesis $G^0 \geq v_n$ in E implies that G has no sinks; thus every vertex in G lies on an infinite path x, and hence in the maximal tail $\gamma := \{v : v \geq x\}$. Thus $G^0 \geq v_n$ says precisely that G^0 is the closure of v_n in D^E. Of course the same is true of w_n in D^F. We can therefore apply Proposition 2.3.3 to deduce that we can by one more standard construction on each of D^E and D^F reach the same 1-sink extension (C, u_n) of D; since all the boundary vertices of D lie in G, this standard construction for extensions of D is a also standard for extensions of G, and hence C can also be obtained by performing n standard constructions to each of E and F.

This completes the proof of the inductive hypothesis, and hence of the theorem.

□

2.5 K-theory of 1-sink extensions

Proposition 2.5.1. *Suppose that G is a row-finite graph with no sinks, and (E, v_0) is a 1-sink extension of G such that $\omega_E \perp \ker(A_G^t - I)$. If F is a 1-sink extension of G and $\phi : C^*(F) \to C^*(E)$ is a $C^*(G)$-embedding, then there exists $k \in \prod_{G^0} \mathbb{Z}$ such that $\omega_E - \omega_F = (A_G - I)k$.*

For the proof, we need to know the K-theory of the C^*-algebras of graphs with sinks, which was was calculated in [78, §3]. We summarize some results from [78] in a convenient form:

Lemma 2.5.2. *Suppose G has no sinks and (E, v_0) is a 1-sink extension of G with graph algebra $C^*(E) = C^*(s_e, p_v)$. Let ψ^E be the homomorphism of $\Big(\bigoplus_{G^0} \mathbb{Z}\Big) \oplus \mathbb{Z}$ into*

$K_0(C^*(E))$ which is determined on the standard basis elements by $\psi^E(\delta_v, 0) := [p_v]$ for $v \in G^0$ and $\psi^E(0,1) = [p_{v_0}]$. Then ψ^E induces an isomorphism of the cokernel of $((A_G^t - I) \oplus \omega_E^t) : \bigoplus_{G^0} \mathbb{Z} \to \left(\bigoplus_{G^0} \mathbb{Z}\right) \oplus \mathbb{Z}$ onto $K_0(C^*(E))$.

Proof. We first suppose that (E, v_0) is simple. Then $\left(\bigoplus_{G^0} \mathbb{Z}\right) \oplus \mathbb{Z}$ is the group $\mathbb{Z}^{G^0} \oplus \mathbb{Z}^W$ considered in [78, §3], and it suffices to show that ψ^E is the homomorphism $\overline{\phi}$ considered there. To do this, we need to check that the map S of $K_0(C^*(E) \times_\gamma \mathbb{T})$ onto $K_0(C^*(E))$ in [78, (3.3)] satisfies $S([p_v \chi_1]) = [p_v]$. The map S is built up from the homomorphisms induced by the embedding of $C^*(E) \times_\gamma \mathbb{T}$ in the dual crossed product $(C^*(E) \times_\gamma \mathbb{T}) \times_{\widehat{\gamma}} \mathbb{Z}$, the Takesaki-Takai duality isomorphism $(C^*(E) \times_\gamma \mathbb{T}) \times_{\widehat{\gamma}} \mathbb{Z} \cong C^*(E) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$, and the map $a \mapsto a \otimes p$ of $C^*(E)$ into $C^*(E) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ determined by a rank-one projection p. The formulas at the start of the proof of [75, Theorem 6] show that, because p_v is fixed under γ, the duality isomorphism carries $p_v \chi_1 \in C^*(E) \times_\gamma \mathbb{T} \subset (C^*(E) \times_\gamma \mathbb{T}) \times_{\widehat{\gamma}} \mathbb{Z}$ into $p_v \otimes M(\chi_1)$, where $M(\chi_1)$ is the projection onto the subspace spanned by the basis element e_1. Thus S has the required property, and the result for simple extensions now follows from [78, Theorem 3.2].

If (E, v_0) is an arbitrary 1-sink extension, we consider its simplification SE and the embedding ϕ^{SE} of $C^*(SE)$ in $C^*(E)$ provided by Proposition 2.1.6, which by [62, Proposition 1.2] induces an isomorphism ϕ_*^{SE} in K-theory. But now it is easy to check that $\phi_*^{SE} \circ \psi^{SE} = \psi^E$, and the result follows. □

We now begin the proof of Proposition 2.5.1. Since the image of ϕ is a full corner in $C^*(E)$, it induces an isomorphism ϕ_* of $K_0(C^*(F))$ onto $K_0(C^*(E))$ (by, for example, [62, Proposition 1.2]). The properties of the $C^*(G)$-embedding ϕ imply that $\phi_*([p_{v_0}]) = [p_{v_0}]$ and $(\pi_E)_* \circ \phi_* = (\pi_F)_*$. We need to know how ϕ_* interacts with the descriptions of K-theory provided by Lemma 2.5.2.

Lemma 2.5.3. *The induced homomorphism $\phi_* : K_0(C^*(F)) \to K_0(C^*(E))$ satisfies $\phi(\psi^F(0,1)) = \psi^E(0,1)$, and for each $z \in \bigoplus_{G^0} \mathbb{Z}$, there exists $\ell \in \mathbb{Z}$ such that $\phi_*(\psi^F(z,0)) = \psi^E(z,\ell)$.*

Proof. The first equation is a translation of the condition $\phi_*([p_{v_0}]) = [p_{v_0}]$. For the second, let $\psi^G : \bigoplus_{G^0} \mathbb{Z} \to K_0(C^*(G))$ be the homomorphism such that $\psi^G(\delta_v) = [p_v]$, which induces the usual isomorphism of $\operatorname{coker}(A_G^t - I)$ onto $K_0(C^*(G))$. If $\rho : (\bigoplus_{G^0} \mathbb{Z}) \oplus \mathbb{Z} \to \bigoplus_{G^0} \mathbb{Z}$ is given by $\rho(z,\ell) := z$, then we have $(\pi_E)_* \circ \psi^E = \psi^G \circ \rho$, and similarly for F. Thus

$$(\pi_E)_* \circ \phi_* \circ \psi^F = (\pi_F)_* \circ \psi^F = \psi^G \circ \rho. \tag{2.4}$$

Now fix $z \in \bigoplus_{G^0} \mathbb{Z}$. Since ψ^E is surjective, there exists $(x,y) \in \left(\bigoplus_{G^0} \mathbb{Z}\right) \oplus \mathbb{Z}$ such that $\psi^E(x,y) = \phi_*(\psi^F(z,0))$. From (2.4) we have

$$\psi^G(z) = (\pi_E)_* \circ \phi_* \circ \psi^F(z,0) = (\pi_E)_* \circ \psi^E(x,y) = \psi^G(x),$$

and hence there exists $u \in \bigoplus_{G^0} \mathbb{Z}$ such that $x = z + (A_G^t - I)u$. Now because ψ^E is constant on the image of $(A_G^t - I) \oplus \omega_E^t$, we have

$$\phi_*(\psi^F(z,0)) = \psi^E(x,y) = \psi^E(z + (A_G^t - I)u, y) = \psi^E(z, y - \omega_E^t u),$$

and $\ell := y - \omega_E^t u$ will do. $\square$

Proof of Proposition 2.5.1. By Lemma 2.5.3, for each $v \in G^0$ there exists $k_v \in \mathbb{Z}$ such that $\phi_*(\psi^F(\delta_v, 0)) = \psi^E(\delta_v, k_v)$. We define $k = (k_v) \in \prod_{G^0} \mathbb{Z}$. A calculation shows that for any $(y, \ell) \in \left(\bigoplus_{G^0} \mathbb{Z}\right) \oplus \mathbb{Z}$ we have

$$\begin{aligned} \phi_* \circ \psi^F(y, \ell) &= \sum_v y_v(\phi_* \circ \psi^F)(\delta_v, 0) + \ell(\phi_* \circ \psi^F)(0,1) \\ &= \left(\sum_v \psi^E(y_v\delta_v, y_v k_v)\right) + \ell\psi^E(0,1) \\ &= \psi^E(y, k^t y + \ell). \end{aligned} \tag{2.5}$$

Now let $z \in \bigoplus_{G^0} \mathbb{Z}$. On one hand, we have from (2.5) that

$$\phi_* \circ \psi^F\Big(((A_G^t - I) \oplus \omega_F^t)(z)\Big) = \psi^E\Big((A_G^t - I)z, k^t(A_G^t - I)z + \omega_F^t z\Big). \tag{2.6}$$

On the other hand, since $\psi^F \circ ((A_G^t - I) \oplus \omega_F^t) = 0$, its composition with ϕ_* is also 0. Thus the class (2.6) must vanish in $K_0(C^*(E))$, and there exists $x \in \bigoplus_{G^0} \mathbb{Z}$ such that

$$\Big((A_G^t - I)z, k^t(A_G^t - I)z + \omega_F^t z\Big) = \Big((A_G^t - I)x, \omega_E^t x\Big). \tag{2.7}$$

Comparing (2.6) and (2.7) shows that $x - z \in \ker(A_G^t - I)$ and

$$k^t(A_G^t - I)z + \omega_F^t z = \omega_E^t x = \omega_E^t z + \omega_E^t(x - z).$$

Since we are supposing $\omega_E \perp \ker(A_G^t - I)$, we deduce that $\omega_E^t(x - z) = 0$. We have now proved that

$$k^t(A_G^t - I)z = (\omega_E^t - \omega_F^t)z \quad \text{for all } z \in \bigoplus\nolimits_{G^0}\mathbb{Z},$$

which implies $(A_G - I)k = \omega_E - \omega_F$, as required. $\square$

Corollary 2.5.4. *Suppose that G is a row-finite graph with no sinks and with the* *property that* $\ker(A_G^t - I) = \{0\}$. *Let* (E_1, v_1) *and* (E_2, v_2) *be 1-sink extensions of* G. *If there is a 1-sink extension* F *such that* $C^*(F)$ *is* $C^*(G)$*-embeddable in both* $C^*(E_1)$ *and* $C^*(E_2)$, *then there exists* $k \in \prod_{G^0} \mathbb{Z}$ *such that* $\omega_{E_1} - \omega_{E_2} = (A_G - I)k$.

Corollary 2.5.5. *Suppose that* G *is a finite graph with no sinks or sources whose vertex matrix* A_G *satisfies* $\ker(A_G^t - I) = 0$. *Let* (E_1, v_1) *and* (E_2, v_2) *be 1-sink extensions of* G *such that* $\overline{v_1} = \overline{v_2}$. *Then there is a 1-sink extension* F *such that* $C^*(F)$ *is* $C^*(G)$*-embeddable in both* $C^*(E_1)$ *and* $C^*(E_2)$ *if and only if there exists* $k \in \bigoplus_{G^0} \mathbb{Z}$ *such that* $\omega_{E_1} - \omega_{E_2} = (A_G - I)k$.

Proof. The forward direction follows from the previous corollary. For the converse, we seek to apply Proposition 2.3.3. To see that n has support in the common closure $C := \overline{v_1} = \overline{v_2}$, recall that A_G decomposes as $A_G = \left(\begin{smallmatrix} B & 0 \\ C & D \end{smallmatrix}\right)$ with respect to $G^0 = (G^0 \setminus C) \cup C$. Thus 1 is an eigenvalue for the $(G^0 \setminus C) \times (G^0 \setminus C)$ corner B of A_G if and only if it is an eigenvalue for A_G, and hence if and only if it is an eigenvalue for A_G^t. So Lemma 2.3.5 applies, $\operatorname{supp} n$ lies in C, and the result follows from Proposition 2.3.3. $\square$

Chapter 3

A primer on the Ext functor

Our goal in this chapter is to give a formal definition of $\mathrm{Ext}(A)$ and to present some useful facts concerning it. We shall begin by stating some definitions and discussing the various notions of equivalence of extensions. After this, we shall define $\mathrm{Ext}(A)$ and establish some of its basic properties. These properties will be useful in the next section where we will prove that the alternate description of $\mathrm{Ext}(A)$ given by Cuntz and Krieger is equivalent to the more familiar definition. Because many of the facts stated here are well known, we will often state them without proof. For a more detailed treatment which includes proofs see [5], [42], or [102].

3.1 Extension preliminaries

Begin by fixing a separable infinite-dimensional Hilbert space $\mathcal{H}$. Throughout this paper we shall let $\mathcal{K}$ denote the compact operators on $\mathcal{H}$, $\mathcal{B}$ denote the bounded operators on $\mathcal{H}$, and $\mathcal{Q} := \mathcal{B}/\mathcal{K}$ be the Calkin algebra. Also let $i : \mathcal{K} \to \mathcal{B}$ denote the inclusion map and $\pi : \mathcal{B} \to \mathcal{Q}$ denote the projection map.

Definition 3.1.1. Let A be a C^*-algebra. An *extension* (j, E, q) of A by $\mathcal{K}$ is a short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{j} E \xrightarrow{q} A \longrightarrow 0.$$

Keep in mind the difference between an extension and the middle C^*-algebra E. If (j, E, q) is an extension of A by $\mathcal{K}$, then there exists a unique homomorphism $\sigma : E \to \mathcal{B}$ such that the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "j"] \arrow[dr, "i"'] & E \arrow[d, "\sigma"] \\ & \mathcal{B} \end{tikzcd}$$

commutes, and a unique homomorphism τ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r, "j"] \arrow[d, equal] & E \arrow[r, "q"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & \mathcal{B} \arrow[r, "\pi"] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

commutes.

Definition 3.1.2. The *Busby invariant* of an extension (j, E, q) is the unique homomorphism τ which makes the above diagram commute.

We call an extension *unital* if E is unital, and *essential* if $j(\mathcal{K})$ is an essential ideal in E. It is a fact that (j, E, q) is unital if and only if its Busby invariant τ is unital, and (j, E, q) is essential if and only if τ is injective. Now suppose that we are given homomorphisms $\alpha_1 : A_1 \to C$ and $\alpha_2 : A_2 \to C$. We may form the *pullback* C^*-algebra

$$E = E(\alpha_1, \alpha_2) := \{ \ (a_1, a_2) \in A_1 \oplus A_2 : \alpha_1(a_1) = \alpha_2(a_2) \ \}.$$

For $i \in \{1, 2\}$ the projection $(a_1, a_2) \mapsto a_i$ from E to A_i will be denoted pr_i. Given any homomorphism $\tau : A \to \mathcal{Q}$ we may form the pullback $E(\tau, \pi)$. If τ is the Busby invariant of an extension (j, E, q), then $\Psi(e) = (q(e), \sigma(e))$ defines an isomorphism from E to $E(\tau, \pi)$ which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r, "j"] \arrow[d, equal] & E \arrow[r, "q"] \arrow[d, "\Psi"] & A \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r, "i_1"] & E(\tau, \pi) \arrow[r, "\mathrm{pr}_1"] & A \arrow[r] & 0
\end{tikzcd}$$

commute. In particular, suppose that (j_1, E_1, q_1) and (j_2, E_2, q_2) are two extensions of A by $\mathcal{K}$ with Busby invariants τ_1 and τ_2, respectively. Then $\tau_1 = \tau_2$ if and only if there exists a homomorphism $\phi : E_1 \to E_2$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r, "j_1"] \arrow[d, equal] & E \arrow[r, "q_1"] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r, "j_2"] & E_2 \arrow[r, "q_2"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes.

Definition 3.1.3. We say that two extensions (j_1, E_1, q_1) and (j_2, E_2, q_2) are *strongly isomorphic* if there exists a homomorphism $\phi : E_1 \to E_2$ such that the above diagram commutes. Note that as a consequence of the three lemma, the homomorphism ϕ must be an isomorphism.

Strong isomorphism induces an equivalence relation on the set of extensions of A by $\mathcal{K}$. We shall denote the set of strong isomorphism classes by $\mathrm{BExt}(A)$. From the preceding discussion of Busby invariants we see that $\mathrm{BExt}(A)$ is in one-to-one correspondence with the set $\mathrm{Hom}(A, \mathcal{Q})$. From this point onward we shall identify an extension (technically the strong isomorphism class of the extension) with its Busby invariant.

Definition 3.1.4. An extension (j, E, q) with Busby invariant τ is *degenerate* if there is a homomorphism $\eta : A \to \mathcal{B}$ such that $\pi \circ \eta = \tau$. In other words, τ can be lifted to a (possibly degenerate) representation η. We call the extension *trivial* if η can be chosen to be nondegenerate; i.e., if $\eta(A)(\mathcal{H})$ is dense in $\mathcal{H}$.

Remark 3.1.5. One should use the above definitions with caution. The terminology given is not standard. What we have called a degenerate extension is more commonly referred to as a trivial extension. However, we have chosen to follow the convention used in [42] and to refer to these extensions as degenerate. We reserve the term trivial for those extensions which can be lifted to nondegenerate homomorphisms.

We now wish to define a binary operation on extensions (or equivalently, the Busby invariants of extensions). Choose an isomorphism $\Theta : M_2(\mathcal{K}) \to \mathcal{K}$. This will induce isomorphisms $\bar{\Theta} : M_2(\mathcal{B}) \to \mathcal{B}$ and $\tilde{\Theta} : M_2(\mathcal{Q}) \to \mathcal{Q}$. Given two extensions with Busby invariants τ_1 and τ_2 we shall let

$$(\tau_1 \oplus \tau_2)(a) = \begin{pmatrix} \tau_1(a) & 0 \\ 0 & \tau_2(a) \end{pmatrix}$$

and we define the sum of τ_1 and τ_2 to be the homomorphism given by

$$(\tau_1 + \tau_2)(a) = \tilde{\Theta}((\tau_1 \oplus \tau_2)(a)).$$

We now enlarge our notion of equivalence of extensions in order to make this operation associative, commutative, and independent of our choice of Θ.

Definition 3.1.6. Two extensions of A by $\mathcal{K}$ with Busby invariants τ_1 and τ_2 are *strongly equivalent* if there exists a unitary $u \in \mathcal{B}$ such that

$$\tau_1 = \mathrm{Ad}(\pi(u)) \circ \tau_2.$$

In this case we shall write $\tau_1 \approx \tau_2$. The set of strong equivalence classes of extensions of A by $\mathcal{K}$ will be denoted $\mathrm{SExt}(A)$ and the class of τ in $\mathrm{SExt}(A)$ will be denoted $[\tau]$.

Proposition 3.1.7. $\mathrm{SExt}(A)$ *with the operation* $[\tau_1] + [\tau_2] = [\tau_1 + \tau_2]$ *is an abelian semigroup.*

Details of this, including a proof of associativity, can be found in [42, Lemma 3.2.3]. Note that the sum of degenerate extensions will be a degenerate extension. Therefore, the degenerate extensions form a subsemigroup of $\mathrm{SExt}(A)$. Hence we may form the quotient, by which we mean the following:

Definition 3.1.8. Two extensions of A by $\mathcal{K}$ with Busby invariants τ_1 and τ_2 are *stably equivalent* if there are degenerate extensions with Busby invariants λ_1 and λ_2, respectively, such that

$$\tau_1 + \lambda_1 \approx \tau_2 + \lambda_2.$$

In this case we write $\tau_1 \sim \tau_2$. The set of stable equivalence classes of extensions of A by $\mathcal{K}$ is denoted $\mathrm{Ext}(A)$, and the class of τ in $\mathrm{Ext}(A)$ will be denoted $[\![\tau]\!]$.

It is not hard to verify that $\mathrm{Ext}(A)$ is an abelian semigroup with identity. Addition is defined by $[\![\tau_1]\!] + [\![\tau_2]\!] := [\![\tau_1 + \tau_2]\!]$, and the identity is the class of any degenerate extension. We shall now define another notion of equivalence which will be of interest to us.

Definition 3.1.9. Two extensions of A by $\mathcal{K}$ with Busby invariants τ_1 and τ_2 are said to be *weakly equivalent* if there is a unitary $u \in \mathcal{Q}$ such that

$$\tau_1 = \mathrm{Ad}(u) \circ \tau_2.$$

In this case we shall write $\tau_1 \approx_w \tau_2$. The set of weak equivalence classes of extensions of A by $\mathcal{K}$ will be denoted $\mathrm{WExt}(A)$ and the class of τ in $\mathrm{WExt}(A)$ will be denoted

$\langle \tau \rangle$.

Weak equivalence turns out to be quite important in the study of extensions. One can prove that the operation

$$\langle \tau_1 \rangle + \langle \tau_2 \rangle := \langle \tau_1 + \tau_2 \rangle$$

is a well-defined operation which makes $\mathrm{WExt}(A)$ into an abelian semigroup. Again, since the sum of two degenerate extensions is degenerate, the degenerate extensions form a subsemigroup and we may form the quotient of $\mathrm{WExt}(A)$ by the degenerate extensions. Formally, this entails the following.

Definition 3.1.10. Two extensions with Busby invariants τ_1 and τ_2 are *weakly stably equivalent* if there are degenerate extensions λ_1 and λ_2 such that

$$\tau_1 + \lambda_1 \approx_w \tau_2 + \lambda_2.$$

In this case we write $\tau_1 \sim_w \tau_2$. The set of weakly stable equivalence classes is denoted $\mathrm{Ext}_w(A)$, and the class of τ in $\mathrm{Ext}_w(A)$ will be denoted $\langle\langle \tau \rangle\rangle$.

It turns out that stable equivalence classes coincide with weak stable equivalence classes; that is, $\mathrm{Ext}(A) = \mathrm{Ext}_w(A)$. Furthermore the essential extensions play an important role.

Definition 3.1.11. Let $\mathrm{WExt}^e(A)$ be the set of weak equivalence classes of essential extensions of A by $\mathcal{K}$. We define $\mathrm{Ext}_w^e(A)$ to be the quotient of $\mathrm{WExt}^e(A)$ by weak stable equivalence. Thus τ_1 and τ_2 give the same class in $\mathrm{Ext}_w^e(A)$ if there are essential degenerate extensions λ_1 and λ_2 such that $\tau_1 + \lambda_1 \approx_w \tau_2 + \lambda_2$.

Proposition 3.1.12. *If there is an essential degenerate extension of A by $\mathcal{K}$, then the natural map of $\mathrm{Ext}_w^e(A)$ into $\mathrm{Ext}_w(A)$ is an isomorphism.*

Remark 3.1.13. If A is a separable C^*-algebra, then there will exist an essential degenerate extension of A by $\mathcal{K}$ [5, §15.5].

This proposition shows us that in most cases we may identify $\mathrm{Ext}_w^e(A)$ with $\mathrm{Ext}_w(A) = \mathrm{Ext}(A)$. Thus we may restrict our attention to only the weak stable equivalence classes of essential extensions. This fact will be important later when we discuss Cuntz and Krieger's description of $\mathrm{Ext}(A)$. We now discuss some useful facts about $\mathrm{Ext}(A)$, beginning with a few deep results due to Voiculescu [99].

Definition 3.1.14. Let ρ and σ be representations of a separable C^*-algebra A on Hilbert spaces $\mathcal{H}_\rho$ and $\mathcal{H}_\sigma$. Then ρ is *approximately equivalent* to σ if there is a sequence $U_n : \mathcal{H}_\rho \to \mathcal{H}_\sigma$ of unitary operators such that for all $a \in A$

1. $U_n \rho(a) U_n^* - \sigma(a)$ is compact, and

2. $\lim_n \|U_n \rho(a) U_n^* - \sigma(a)\| = 0$.

Voiculescu's results have been summarized in a nice article by Arveson [2]. We give Arveson's formulation of Voiculescu's Theorem here [2, Corollary 1 to Theorem 5].

Theorem 3.1.15 (Voiculescu). *Let ρ and σ be nondegenerate representations of a separable C^*-algebra A.*

1. *If $\ker \rho = \ker \sigma$ and neither $\rho(A)$ nor $\sigma(A)$ contains any compact operators, then ρ is approximately equivalent to σ.*

2. *If σ vanishes on $\ker \pi \circ \sigma$, then $\rho \oplus \sigma$ is approximately equivalent to ρ.*

Using this result it is possible to prove a "uniqueness" result for degenerate extensions. Since unital extensions and nonunital extensions cannot be even weakly equivalent there are two cases.

Lemma 3.1.16. *Suppose that A is separable. Any two essential trivial extensions are strongly equivalent. Similarly, any two essential nonunital degenerate extensions of A by $\mathcal{K}$ are strongly equivalent.*

Recall that strong equivalence implies weak equivalence, so the above statement is also true for weak equivalence. In addition, from Voiculescu's theorem we may obtain results concerning absorbing and unital-absorbing extensions.

Definition 3.1.17. An extension of A by $\mathcal{K}$ with Busby invariant τ is said to be *absorbing* if τ is strongly equivalent to $\tau + \lambda$ for every degenerate extension λ. If A is unital and τ is unital, then we say that τ is *unital-absorbing* if τ is strongly equivalent to $\tau + \sigma$ for every trivial extension σ. (Note that σ is necessarily unital.)

Lemma 3.1.18. *Suppose that A is separable. Then every essential nonunital extension of A by $\mathcal{K}$ is absorbing. If A is unital, then every essential unital extension of A by $\mathcal{K}$ is unital-absorbing.*

See [2] for a good exposition and proof of the above results.

3.2 Cuntz and Krieger's description of Ext(A)

In [15] Cuntz and Krieger computed $\operatorname{Ext} \mathcal{O}_A$ using a slightly nonstandard description of Ext. We will want to make use of this description, so in this section we give an expanded version of it and prove that in general it is equivalent to the usual definition. Just as this description was useful for computing Ext for Cuntz-Krieger algebras, we shall see in Chapter 4 that it is useful for computing Ext for graph algebras. In addition, it is the author's belief that this description is also of independent interest. This is because the description applies to arbitrary C^*-algebras and not just graph algebras. Hence it gives a more tractable way of determining the equivalence classes which make up Ext.

We begin by mentioning that Cuntz and Krieger work exclusively with essential extensions. Therefore we shall suppose that there exists an essential degenerate extension of A by $\mathcal{K}$, and identify $\mathrm{Ext}(A)$ with $\mathrm{Ext}_w^e(A)$ courtesy of Proposition 3.1.12 and the fact that $\mathrm{Ext}(A) = \mathrm{Ext}_w(A)$.

Definition 3.2.1. We say that two Busby invariants τ_1 and τ_2 are *CK-equivalent* if there exists a partial isometry $v \in \mathcal{Q}$ such that

$$\tau_1 = \mathrm{Ad}(v) \circ \tau_2 \quad \text{and} \quad \tau_2 = \mathrm{Ad}(v^*) \circ \tau_1. \tag{3.1}$$

Remark 3.2.2. Note that CK-equivalence is clearly reflexive and symmetric. However, because v is a partial isometry it is not obvious whether it is also transitive. In the following lemma we show that two essential extensions are CK-equivalent if and only if they are weakly stably equivalent. Hence CK-equivalence is transitive for essential extensions. It is unclear to the author at this time whether CK-equivalence is transitive in general.

Lemma 3.2.3. *Suppose that τ_1 and τ_2 are the Busby invariants of two essential extensions of A by $\mathcal{K}$. Then τ_1 equals τ_2 in $\mathrm{Ext}_w^e(A)$ if and only if τ_1 and τ_2 are CK-equivalent.*

Before giving the proof we need a couple of observations.

Lemma 3.2.4. *Suppose that λ_1 and λ_2 are essential degenerate extensions of A by $\mathcal{K}$. Then there is a partial isometry $v \in \mathcal{Q}$ such that*

$$\lambda_1 = \mathrm{Ad}(v) \circ \lambda_2 \qquad \lambda_2 = \mathrm{Ad}(v^*) \circ \lambda_1.$$

Thus all essential degenerate extensions are CK-equivalent.

Proof. Let $\lambda_i = \pi \circ \hat{\lambda}_i$ for possibly degenerate representations $\hat{\lambda}_i : A \to \mathcal{B}(\mathcal{H})$. Since the $\hat{\lambda}_i$ may be degenerate representations, we can't apply Voiculescu's Theorem directly. However, recall that

$$\mathcal{H}_i := \overline{\text{span}}\{\hat{\lambda}_i(a)h : a \in A \text{ and } h \in \mathcal{H}\}$$

is an invariant subspace for $\hat{\lambda}_i$ and is called the *essential subspace* of $\hat{\lambda}_i$. The restriction ess $\hat{\lambda}_i$ of $\hat{\lambda}_i$ to $\mathcal{H}_i$ is nondegenerate and is called the essential part of $\hat{\lambda}_i$. Note that $\hat{\lambda}_i = \text{ess } \hat{\lambda}_i \oplus 0$. Since each λ_i is faithful by assumption, ess $\hat{\lambda}_i$ is injective and ess $\hat{\lambda}_i(A) \cap \mathcal{K}(\mathcal{H}_i) = 0$. Now Voiculescu's Theorem implies that there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$(\text{ess } \hat{\lambda}_2)(a) - U(\text{ess } \hat{\lambda}_1)(a)U^*$$

is compact for all $a \in A$. It follows that there is a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that both

$$\hat{\lambda}_2(a) - V\hat{\lambda}_1(a)V^* \quad \text{and} \quad \hat{\lambda}_1(a) - V^*\hat{\lambda}_2(a)V$$

are compact for all $a \in A$. The result follows. □

Lemma 3.2.5. *Let τ_1 and τ_2 be extensions which are CK-equivalent. If $\tau_1' \approx_w \tau_1$ and $\tau_2' \approx_w \tau_2$, then τ_1' and τ_2' are CK-equivalent.*

Proof. Straightforward. □

Proof of Lemma 3.2.3. Suppose that there exists a partial isometry $v \in \mathcal{Q}$ for which (3.1) holds. If $v^*v = vv^* = 1$, then $\tau_1 \approx_w \tau_2$, and we trivially have $\tau_1 \sim_w \tau_2$. If $v^*v < 1$ and $vv^* < 1$, then there is a partial isometry u with $u^*u = 1 - v^*v$ and

$uu^* = 1 - vv^*$. Then $u + v$ is a unitary, and

$$(v+u)\tau_2(a)(v^*+u^*) = (v+u)v^*v\tau_2(a)v^*v(v^*+u^*) = v\tau_2(a)v^* = \tau_1(a)$$

so $\tau_1 \approx_w \tau_2$ and we again have $\tau_1 \sim_w \tau_2$. So we may as well assume that v is a nonunitary isometry; i.e. $v^*v = 1$ and $vv^* < 1$. Let $\sigma_1 = \pi \circ \hat{\sigma}_1$ be an essential degenerate extension, and let σ_2 be the essential degenerate extension coming from the compression of $\hat{\sigma}_1$, i.e. $\sigma_2 := \pi \circ \mathrm{Ad}(v) \circ \hat{\sigma}_1$. Let $U \in M_2(\mathcal{Q})$ be given by

$$U = \begin{pmatrix} v^* & 0 \\ 1 - vv^* & v \end{pmatrix}.$$

Then U is a unitary with

$$U^{-1} = U^* = \begin{pmatrix} v & 1 - vv^* \\ 0 & v^* \end{pmatrix}$$

and $U(\tau_1 \oplus \sigma_1)U^* = \tau_2 \oplus \sigma_2$. It follows that $\tau_1 + \sigma_1 \approx_w \tau_2 + \sigma_2$, and $\tau_1 \sim_w \tau_2$. Thus we have shown that (3.1) implies weak stable equivalence.

Now assume that $\tau_1 \sim_w \tau_2$. Suppose that $\tau_1 + \lambda_1 \approx_w \tau_2 + \lambda_2$ with each λ_i degenerate, and $\lambda_i = \pi \circ \hat{\lambda}_i$. If both τ_1 and τ_2 are nonunital, then both are absorbing by [5, Theorem 15.12.3]. Consequently, $\tau_1 \approx_w \tau_2$ and (3.1) certainly holds. Now suppose that τ_1 is unital and τ_2 is not. Let v be a nonunitary isometry in $\mathcal{Q}$, and define

$$\tau_1'(a) := v\tau_1(a)v^* \text{ for all } a \in A.$$

Since τ_1 and τ_1' are CK-equivalent, $\tau_1' + \sigma_1 \approx_w \tau_1 + \sigma_2$, where σ_1 and σ_2 are as above. Since $\sim_w$ is transitive, $\tau_1' \sim_w \tau_2$. Furthermore, because neither τ_1' nor τ_2 is unital we know that they are absorbing by Lemma 3.1.18 and thus there is a unitary u such that $\tau_1' = \mathrm{Ad}(u) \circ \tau_2$. Thus

$$\tau_2(a) = u^* v \tau_1(a) v^* u \quad \text{and} \quad \tau_1(a) = v^* u \tau_2(a) u^* v.$$

Because u is a unitary, it follows that u^*v is a partial isometry and (3.1) holds. It only remains to consider the case that both τ_i's are unital. We let u be a unitary in $\mathcal{Q}$ such that

$$u(\tau_1 + \lambda_1)u^* = \tau_2 + \lambda_2.$$

Let λ be a degenerate extension of A by $\mathcal{K}$ which lifts to a unital homomorphism. Since Lemma 3.1.18 implies that each τ_i is unital-absorbing, it follows that $\tau_1 + \lambda \approx \tau_1$ and $\tau_2 \approx \tau_2 + \lambda$. Thus it suffices to show that $\tau_1 + \lambda$ is CK-equivalent to $\tau_2 + \lambda$. To do this, notice by Lemma 3.2.4 there are isometries $w_i \in \mathcal{Q}$ such that

$$\lambda = \mathrm{Ad}(w_i^*) \circ \lambda_i.$$

It follows that there are isometries $v_i \in \mathcal{Q}$ such that

$$\tau_i + \lambda = \mathrm{Ad}(v_i^*) \circ (\tau_i + \lambda_i).$$

Notice that

$$w_i w_i^* = (\tau_i + \lambda_i)(1) \quad \text{and} \quad u((\tau_1 + \lambda_1)(1))u^* = (\tau_2 + \lambda_2)(1).$$

Now we compute that

$$\tau_1 + \lambda = \mathrm{Ad}(v_1^*) \circ (\tau_1 + \lambda_1) = \mathrm{Ad}(v_1^* u^*) \circ (\tau_2 + \lambda_2) = \mathrm{Ad}(v_1^* u^* v_2) \circ (\tau_2 + \lambda).$$

Therefore it will suffice to observe that $v_1^*u^*v_2$ is a coisometry. But

$$v_1^*u^*v_2v_2^*uv_1 = v_1^*u^*((\tau_2+\lambda)(1))uv_1 = v_1^*((\tau_1+\lambda_1)(1))v_1 = v_1^*v_1v_1^*v_1 = 1. \qquad \square$$

In light of this lemma we may think of the class of τ in $\mathrm{Ext}_w^e(A)$ as the class generated by the relation in (3.1). Furthermore, we see that any two essential degenerate extensions will be equivalent.

For extensions τ_1 and τ_2 we say that $\tau_1 \perp \tau_2$ if there are orthogonal projections p_1 and p_2 such that $\tau_i(A) \subseteq p_i\mathcal{Q}p_i$. In this case we may define a map $\tau_1 \boxplus \tau_2$ by $a \mapsto \tau_1(a) + \tau_2(a)$. The orthogonality of the projections is enough to ensure that this map will be multiplicative and therefore $\tau_1 \boxplus \tau_2$ will be a homomorphism. The notation $\boxplus$ is used because a quite different meaning has already been assigned to $\tau_1 + \tau_2$.

Now suppose that $\langle\!\langle\tau_1\rangle\!\rangle, \langle\!\langle\tau_2\rangle\!\rangle \in \mathrm{Ext}_w^e(A)$. We may choose two isometries $v_1, v_2 \in \mathcal{B}(\mathcal{H})$ with orthogonal ranges. If we define $\tau_i' = \pi(v_i)\tau_i\pi(v_i)^*$, then τ_1' and τ_2' are homomorphisms with $\tau_1' \perp \tau_2'$. Furthermore, $\tau_1' \oplus \tau_2'$ will be unitarily equivalent to $\tau_1' \boxplus \tau_2' \oplus 0$. Consequently $[\![\tau_1' + \tau_2']\!] = [\![\tau_1' \boxplus \tau_2']\!]$. Since stable equivalence classes coincide with weak stable equivalence classes, it follows that $\langle\!\langle\tau_1' + \tau_2'\rangle\!\rangle = \langle\!\langle\tau_1' \boxplus \tau_2'\rangle\!\rangle$. Furthermore, since τ_i' is CK-equivalent to τ_i it follows from Lemma 3.2.3 that $\tau_i' \sim_w \tau_i$. Thus $\langle\!\langle\tau_1 + \tau_2\rangle\!\rangle = \langle\!\langle\tau_1' \boxplus \tau_2'\rangle\!\rangle$.

This gives us Cuntz and Krieger's description of $\mathrm{Ext}(A)$. Provided that there exists an essential degenerate extension of A by $\mathcal{K}$, we may identify $\mathrm{Ext}(A)$ with $\mathrm{Ext}_w^e(A)$ which we then view as the equivalence classes of essential extensions τ generated by the relation in (3.1). For elements $\langle\!\langle\tau_1\rangle\!\rangle, \langle\!\langle\tau_2\rangle\!\rangle \in \mathrm{Ext}_w^e(A)$, we define their sum to be $\langle\!\langle\tau_1\rangle\!\rangle + \langle\!\langle\tau_2\rangle\!\rangle = \langle\!\langle\tau_1' \boxplus \tau_2'\rangle\!\rangle$ where τ_1' and τ_2' are essential extensions such that $\tau_1' \perp \tau_2'$

and $\tau_i' \sim_w \tau_i$. Note that the common class of all degenerate essential extensions acts as the neutral element in $\mathrm{Ext}_w^e(A)$.

Chapter 4

Computing Ext for graph algebras

The role of Ext in the study of Cuntz-Krieger algebras has a long history. As early as 1978 Cuntz algebras were classified by Paschke and Salinas using Ext [63] and this was also done simultaneously by Pimsner and Popa [74]. In Cuntz and Krieger's seminal paper [15] Ext was computed for Cuntz-Krieger algebras. If $\mathcal{O}_A$ is the Cuntz-Krieger algebra associated to an $n \times n$ matrix A, then Cuntz and Krieger showed that $\operatorname{Ext}(\mathcal{O}_A)$ is isomorphic to $\operatorname{coker}(A - I)$, where $A : \mathbb{Z}^n \to \mathbb{Z}^n$. In this chapter we extend Cuntz and Krieger's computation of $\operatorname{Ext} \mathcal{O}_A$ to graph algebras. Specifically, we prove the following.

Theorem. *Let G be a row-finite graph with no sinks and in which every loop has an exit, and let $C^*(G)$ be the C^*-algebra associated to G. Then there exists an isomorphism*

$$\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$$

where A_G is the vertex matrix of G and $A_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$.

In addition to showing that $\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(A_G - I)$, the isomorphism ω is important because its value on certain extensions can be easily calculated. If E

is an essential 1-sink extension of G as described in Chapter 2, then $C^*(E)$ will be an extension of $C^*(G)$ by $\mathcal{K}$ and thus determines an element in $\operatorname{Ext}(C^*(G))$. For example, if G is the graph

$$G \qquad \begin{tikzcd} w_1 \arrow[loop above] \arrow[r] & w_2 \arrow[r] & w_3 \arrow[l, bend right] \end{tikzcd}$$

then two examples of essential 1-sink extensions are the following graphs E_1 and E_2:

$$E_1 \qquad \begin{tikzcd} w_1 \arrow[loop above] \arrow[r] \arrow[dr] & w_2 \arrow[r] \arrow[d] & w_3 \arrow[l, bend right] \arrow[dl] \arrow[dl, bend left] \\ & v_0 & \end{tikzcd} \qquad E_2 \qquad \begin{tikzcd} w_1 \arrow[loop above] \arrow[r] \arrow[dr] & w_2 \arrow[r] & w_3 \arrow[l, bend right] \arrow[dl] \\ & v_0 & \end{tikzcd}$$

In the above two examples the Wojciech vector is the vector whose v^{th} entry is equal to the number of edges from v to the sink. This vector is $\left(\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right)$ for E_1 and $\left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$ for E_2. It turns out that if E is a 1-sink extension of G, then the value that ω assigns to the element of $\operatorname{Ext}(C^*(G))$ associated to E is equal to the class of the Wojciech vector of E in $\operatorname{coker}(A_G - I)$. Furthermore, since ω is additive we have a nice way of describing addition of elements in $\operatorname{Ext}(C^*(G))$ associated to essential 1-sink extensions. For example, if E_1 and E_2 are as above, then the sum of their associated elements in $\operatorname{Ext}(C^*(G))$ is the element in $\operatorname{Ext}(C^*(G))$ associated to the 1-sink extension

$$\begin{tikzcd} w_1 \arrow[loop above] \arrow[r] \arrow[dr] \arrow[dr, bend right] & w_2 \arrow[r] \arrow[d] & w_3 \arrow[l, bend right] \arrow[dl] \arrow[dl, bend left] \arrow[dl, bend right] \\ & v_0 & \end{tikzcd}$$

whose Wojciech vector is $\left(\begin{smallmatrix}2\\1\\3\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right) + \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$. Thus we have a way of realizing certain elements of $\operatorname{Ext}(C^*(G))$ as well as a way to visualize their sums. We show in §4.2

that if G is a finite graph, then every element of $\mathrm{Ext}(C^*(G))$ is an element associated to an essential 1-sink extension of G. We also show that this is not necessarily the case for infinite graphs.

In addition to providing an easily visualized description of $\mathrm{Ext}(C^*(G))$, we also show that the isomorphism ω can be used to ascertain information about the semiprojectivity of a graph algebra. Blackadar has shown that the Cuntz-Krieger algebras are semiprojective [6], and Szymański has proven that C^*-algebras of transitive graphs with finitely many vertices are semiprojective [92]. Although not all graph algebras are semiprojective (for instance, it follows from [6, Theorem 3.1] that $\mathcal{K}$ is not semiprojective), it is natural to wonder if the C^*-algebras of transitive graphs will always be semiprojective. In §4.3 we answer this question in the negative. We use the isomorphism ω to produce an example of a row-finite transitive graph whose C^*-algebra is not semiprojective.

This chapter is organized as follows. We begin in §4.1 by creating a map $d : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(B_G - I)$, where B_G is the edge matrix of G. In §4.2 we define the map $\omega : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(A_G - I)$, where A_G is the vertex matrix of G. We also prove that ω is an isomorphism and compute the value it assigns to elements of $\mathrm{Ext}(C^*(G))$ associated to essential 1-sink extensions. We conclude in §4.3 by providing an example of a row-finite transitive graph whose C^*-algebra is not semiprojective. Throughout we shall make use of Cuntz and Krieger's description of Ext discussed in §3.2.

4.1 The Ext group for $C^*(G)$

Throughout this chapter we shall let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space, $\mathcal{K}$ denote the compact operators on $\mathcal{H}$, $\mathcal{B}$ denote the bounded operators on $\mathcal{H}$, and $\mathcal{Q} := \mathcal{B}/\mathcal{K}$ denote the Calkin algebra. We shall also let $i : \mathcal{K} \to \mathcal{B}$ denote the inclusion map and $\pi : \mathcal{B} \to \mathcal{Q}$ denote the projection map.

Lemma 4.1.1. *Suppose that $p_1, p_2, \ldots$ is a countable sequence of pairwise orthogonal projections in $\mathcal{Q}$. Then there are pairwise orthogonal projections $P_1, P_2, \ldots$ in $\mathcal{B}$ such that $\pi(P_i) = p_i$ for $i = 1, 2, \ldots$.*

Proof. We first show how to find $P_1 \in \mathcal{B}$ such that $\pi(P_1) = p_1$. Lift p to a self-adjoint element $T \in \mathcal{B}$. Then $\pi(T^2 - T) = 0$ and $T^2 - T$ is compact. Therefore $\sigma(T^2 - T)$ is a countable set whose only accumulation point is 0. By the functional calculus we know that $\sigma(T^2 - T) = \{a^2 - a : a \in \sigma(T)\}$. Because of the continuity of $f(x) = x^2 - x$ we see that 0 is also the only accumulation point of $\sigma(T)$. In particular, there exists $a \in (0, 1)$ and $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \cap \sigma(T) = \emptyset$. Thus we may use functional calculus to create P_1.

We now give a recursive definition for the other P_i's. Suppose that $P_1, \ldots, P_n$ are pairwise orthogonal lifts of $p_1, \ldots, p_n$ to projections in $\mathcal{B}$. Let P'_{n+1} be any lift of p_{n+1} to a projection. Let $P''_{n+1} := (1 - P_1 - \ldots - P_n)P'_{n+1}(1 - P_1 - \ldots - P_n)$. Then $\pi(P''_{n+1}) = p_{n+1}$, $P''_{n+1}P_i = 0$ if $i = 1, 2, \ldots, n$, and P''_{n+1} is self-adjoint. As in the previous paragraph, we may use the functional calculus to obtain a continuous function f for which $P_{n+1} := f(P''_{n+1})$ is a projection, $f(0) = 0$, and $\pi(P_{n+1}) = p_{n+1}$. Since P_{n+1} can be approximated by polynomials in P''_{n+1} with no constant terms, it follows that $P_{n+1}P_i = 0$ for all $1 \leq i \leq n$. Taking adjoints implies that $P_iP_{n+1} = 0$ as well. $\square$

The following comes from [16, Lemma V.6.4].

Lemma 4.1.2. *If w is a partial isometry in $\mathcal{Q}$, then there exists a partial isometry V in $\mathcal{B}$ such that $\pi(V) = w$.*

Lemma 4.1.3. *If w is a unitary in $\mathcal{Q}$, then w can be lifted to either an isometry or coisometry $U \in \mathcal{B}$.*

Proof. By Lemma 4.1.2 we may choose a partial isometry $V \in \mathcal{B}$ for which $\pi(V) = w$. Let $P = V^*V$ and $Q = VV^*$. Because $1 - P$ and $1 - Q$ are compact projections, it follows that $1 - P$ and $1 - Q$ have finite rank. Replacing w by w^* if necessary, we may assume that rank $(1 - P) \leq$ rank $(1 - Q)$. Let V_0 be any partial isometry with source projection $V_0^*V_0 = 1 - P$ and range projection $V_0V_0^* \leq 1 - Q$. Then $U := V + V_0$ is an isometry and since $V_0 = V_0V_0^*V_0 = V_0(1 - P)$ is compact, we see that $\pi(U) = \pi(V + V_0) = \pi(V) = w$. □

For the rest of this section let G be a row-finite graph with no sinks that satisfies Condition (L). Since $C^*(G)$ is separable, there will exist an essential degenerate extension of $C^*(G)$ [5, §15.5]. (In fact, we shall prove that there are many essential degenerate extensions in Lemma 4.1.7.) Therefore we may use Cuntz and Krieger's description of Ext discussed in §3.2.

Let $E \in \mathcal{Q}$ be a projection. By Lemma 4.1.1 we know that there exists a projection $E' \in \mathcal{B}$ such that $\pi(E') = E$. If X is an element of $\mathcal{Q}$ such that EXE is invertible in $E\mathcal{Q}E$, then we denote by $\operatorname{ind}_E X$ the Fredholm index of $E'X'E'$ in $\operatorname{im} E'$, where $X' \in \mathcal{B}$ is such that $\pi(X') = X$. Since the Fredholm index is invariant under compact perturbations [11, Theorem XI.3.11], this definition does not depend on the choice of E' or X'. The following two lemmas are taken from [15] where no proofs are given.

Lemma 4.1.4. *Let $E, F \in \mathcal{Q}$ be orthogonal projections, and let X be an element of $\mathcal{Q}$ such that EXE and FXF are invertible in $E\mathcal{Q}E$ and $F\mathcal{Q}F$ and such that X commutes with E and F. Then* $\operatorname{ind}_{E+F} X = \operatorname{ind}_E X + \operatorname{ind}_F X$.

Proof. Let $\operatorname{Fred}_{\mathcal{H}} T$ denote the Fredholm index of T in $\mathcal{H}$. Since E and F are orthogonal projections in $\mathcal{Q}$ we may use Lemma 4.1.1 to find orthogonal projections $E', F' \in \mathcal{B}$ with $\pi(E') = E$ and $\pi(F') = F$. Now since E' is orthogonal to F' we see that $\ker(E'X'E' + F'X'F')$ in $\operatorname{im}(E' + F')$ is isomorphic to the direct sum of $\ker E'X'E'$ in $\operatorname{im} E'$ and $\ker F'X'F'$ in $\operatorname{im} F'$. Similarly for the kernel of $(E'X'E' + F'X'F')^* = (E'X'^*E' + F'X'^*F')$. Thus

$$\mathrm{Fred}_{\operatorname{im}(E'+F')}(E'X'E' + F'X'F') = \mathrm{Fred}_{\operatorname{im} E'} E'X'E' + \mathrm{Fred}_{\operatorname{im} F'} F'X'F' \tag{4.1}$$

Now since $\pi(E'X'F') = EXF = EFX = 0$ we see that $E'X'F'$ is a compact operator. Similarly $F'X'E'$ is a compact operator. Because Fredholm index is unchanged by compact perturbations [11, Theorem XI.3.11] we have

$$\begin{aligned}\mathrm{Fred}_{\operatorname{im}(E'+F')}&(E'X'E' + E'X'F' + F'X'E' + F'X'F') \\ &= \mathrm{Fred}_{\operatorname{im}(E'+F')}(E'X'E' + F'X'F'). \end{aligned}\tag{4.2}$$

Now since

$$E'X'E' + E'X'F' + F'X'E' + F'X'F' = (E' + F')X'(E' + F')$$

Eq. 4.1 and Eq. 4.2 show that

$$\mathrm{Fred}_{\operatorname{im}(E'+F')}(E' + F')X'(E' + F') = \mathrm{Fred}_{\operatorname{im} E'} E'X'E' + \mathrm{Fred}_{\operatorname{im} F'} F'X'F'$$

and thus $\operatorname{ind}_{E+F} X = \operatorname{ind}_E X + \operatorname{ind}_F X$. □

Lemma 4.1.5. *Let X and Y be invertible operators in EQE. Then $\operatorname{ind}_E XY = \operatorname{ind}_E X + \operatorname{ind}_E Y$.*

Proof. Let $\mathrm{Fred}_{\mathcal{H}} T$ denote the Fredholm index of T in $\mathcal{H}$. Let E' be a projection in $\mathcal{B}$ with $\pi(E') = E$, and let $X', Y' \in \mathcal{B}$ with $\pi(X') = X$ and $\pi(Y') = Y$. Now since

X and Y are operators in EQE we see that $EXYE = (EXE)(EYE)$. Therefore $\pi((E'X'E')(E'Y'E') - E'X'Y'E') = 0$ and $(E'X'E')(E'Y'E') - E'X'Y'E'$ is a compact operator. Now since Fredholm index is unchanged by compact perturbations [11, Theorem XI.3.11] we have

$$\begin{aligned}\operatorname{Fred}_{\operatorname{im} E'} E'X'Y'E' &= \operatorname{Fred}_{\operatorname{im} E'} E'X'Y'E' + (E'X'E')(E'Y'E') - E'X'Y'E' \\ &= \operatorname{Fred}_{\operatorname{im} E'} (E'X'E')(E'Y'E')\end{aligned}$$

and by [11, Theorem XI.3.7] that

$$\operatorname{Fred}_{\operatorname{im} E'} (E'X'E')(E'Y'E') = \operatorname{Fred}_{\operatorname{im} E'} E'X'E' + \operatorname{Fred}_{\operatorname{im} E'} E'Y'E'.$$

Hence $\operatorname{Fred}_{\operatorname{im} E'} E'X'Y'E' = \operatorname{Fred}_{\operatorname{im} E'} E'X'E' + \operatorname{Fred}_{\operatorname{im} E'} E'Y'E'$ and $\operatorname{ind}_E XY = \operatorname{ind}_E X + \operatorname{ind}_E Y$. $\square$

In addition, we shall make use of the following lemmas to define a map from $\operatorname{Ext}(C^*(G))$ into $\operatorname{coker}(B_G - I)$. The first lemma is an immediate consequence of the Gauge-Invariant Uniqueness Theorem for graph algebras [4, Theorem 3.1].

Lemma 4.1.6. *Let G be a graph that satisfies Condition (L), and let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger G-family in $C^*(G)$. If I is an ideal of $C^*(G)$ with the property that $p_v \notin I$ for all $v \in G^0$, then $I = \{0\}$.*

Lemma 4.1.7. *Let G be a row-finite graph with no sinks that satisfies Condition (L), and let $\tau : C^*(G) \to \mathcal{Q}$ be an essential extension of $C^*(G)$. If $\{s_e, p_v\}$ is the canonical Cuntz-Krieger G-family, then there exists a degenerate essential extension $t : C^*(G) \to \mathcal{Q}$ such that $t(s_e s_e^*) = \tau(s_e s_e^*)$ for all $e \in G^1$.*

Proof. Since τ is essential, $\{\tau(s_e s_e^*)\}_{e \in G^1}$ is a countable set of mutually orthogonal nonzero projections and we may use Lemma 4.1.1 to lift them to a collection $\{R_e\}_{e\in G^1}$ of mutually orthogonal nonzero projections in $\mathcal{B}$. Now each $\mathcal{H}_e := \text{im } R_e$ is infinite-dimensional, and for each $v \in G^0$ we define $\mathcal{H}_v = \bigoplus_{\{s(e)=v\}} \mathcal{H}_e$. Then each $\mathcal{H}_v$ is infinite-dimensional and for each $e \in G^1$ we can let T_e be a partial isometry with initial space $\mathcal{H}_{r(e)}$ and final space $\mathcal{H}_e$. Also for each $v \in G^0$ we shall let Q_v be the projection onto $\mathcal{H}_v$. Then $\{T_e, Q_v\}$ is a Cuntz-Krieger G-family. By the universal property of $C^*(G)$ there exists a homomorphism $\tilde{t} : C^*(G) \to \mathcal{B}$ such that $\tilde{t}(p_v) = Q_v$ and $\tilde{t}(s_e) = T_e$. Let $t := \pi \circ \tilde{t}$. Then t is a degenerate extension and $t(s_e s_e^*) = \pi(\tilde{t}(s_e s_e^*)) = \pi(T_e T_e^*) = \pi(R_e) = \tau(s_e s_e^*)$. Furthermore, for all $v \in G^0$ we have that

$$t(p_v) = \sum_{s(e)=v} t(s_e s_e^*) = \sum_{s(e)=v} \tau(s_e s_e^*) = \tau(p_v) \neq 0,$$

so t is essential. $\square$

Remark 4.1.8. Suppose that G is a graph with no sinks, τ is an extension of $C^*(G)$, and t is another extension for which $t(s_e s_e^*) = \tau(s_e s_e^*)$. Then t will also have the property that $t(p_v) = t(\sum s_e s_e^*) = \sum t(s_e s_e^*) = \sum \tau(s_e s_e^*) = \tau(\sum s_e s_e^*) = \tau(p_v)$ for any $v \in G^0$.

Definition 4.1.9. Let $\tau : C^*(G) \to \mathcal{Q}$ be an essential extension of $C^*(G)$, and for each $e \in G^1$ define $E_e := \tau(s_e s_e^*)$. If $t : C^*(G) \to \mathcal{Q}$ is another essential extension of $C^*(G)$ with the property that $t(s_e s_e^*) = E_e$, then we define a vector $d_{\tau,t} \in \prod_{G^1} \mathbb{Z}$ by

$$d_{\tau,t}(e) = -\operatorname{ind}_{E_e} \tau(s_e) t(s_e^*).$$

Note that this is well-defined since $E_e \tau(s_e) t(s_e^*) E_e = \tau(s_e) t(s_e^*)$ and by Remark 4.1.8 we have that $\tau(s_e) t(s_e^*) \tau(s_e^*) t(s_e) = \tau(s_e) \tau(s_e^* s_e) \tau(s_e^*) = E_e$ so $\tau(s_e) t(s_e^*)$ is invertible in $E_e \mathcal{Q} E_e$.

Remark 4.1.10. If $E \in \mathcal{Q}$ is a projection and $E' \in \mathcal{B}$ is a lift of E to a projection in $\mathcal{B}$, then one can see that $\mathcal{Q}(E'(\mathcal{H})) \cong E\mathcal{Q}E$ via the obvious correspondence. In the rest of this paper we shall often identify $\mathcal{Q}(E'(\mathcal{H}))$ with $E\mathcal{Q}E$.

Lemma 4.1.11. *Let $E \in \mathcal{Q}$ be a projection and $X \in \mathcal{Q}$, and suppose that EXE is invertible in $E\mathcal{Q}E$. If $V \in \mathcal{Q}$ is a partial isometry with initial projection $V^*V = E$ and final projection $VV^* = F$, then* $\operatorname{ind}_E X = \operatorname{ind}_F VXV^*$.

Proof. By Lemma 4.1.2 there exists a partial isometry $W \in \mathcal{B}$ such that $\pi(W) = V$. Let $E' := W^*W$ and $F' := WW^*$ Also choose any $X' \in \mathcal{B}$ such that $\pi(X') = X$. Define $\Phi_V : \mathcal{B}(E'(\mathcal{H})) \to \mathcal{B}(F'(\mathcal{H}))$ by

$$\Phi_V(T) = WTW^*.$$

We shall show that Φ_V is a homomorphism. To see that Φ_V is injective let $\Phi_V(T) = 0$. Then

$$T = E'TE' = W^*WTW^*W = W^*\Phi_V(T)W = 0$$

and thus $\ker \Phi_V = \{0\}$.

In addition, if $T \in \mathcal{B}(F'(\mathcal{H}))$, then

$$E'(W^*FW)E' = W^*WW^*FWW^*W = W^*FW$$

and $W^*FW \in \mathcal{B}(E'(\mathcal{H}))$. Because $\Phi_V(W^*FW) = F$ this implies that Φ_V is surjective. Therefore, $\Phi_V : \mathcal{B}(E'(\mathcal{H})) \to \mathcal{B}(F'(\mathcal{H}))$ is an isomorphism.

Because Φ_V is an isomorphism, the fact that EXE is an invertible operator in $\mathcal{Q}(E'(\mathcal{H})E') \cong E\mathcal{Q}E$ implies that the Fredholm index of $E'XE'$ in $\mathcal{B}(E'(\mathcal{H}))$ is the same as the Fredholm index of $\Phi_V(E'XE') = FWXW^*F'$ in $\mathcal{B}(F'(\mathcal{H}))$. Therefore $\operatorname{ind}_E X = \operatorname{ind}_F VXV^*$. □

Proposition 4.1.12. *Let G be a row-finite graph with no sinks that satisfies Condition (L). Also let τ be an essential extension of $C^*(G)$ and $E_e := \tau(s_e s_e^*)$ for $e \in G^1$. If t and t' are essential extensions of $C^*(G)$ that are CK-equivalent and satisfy $t(s_e s_e^*) = t'(s_e s_e^*) = E_e$, then $d_{\tau,t} - d_{\tau,t'} \in \operatorname{im}(B_G - I)$.*

Proof. Since t and t' are CK-equivalent, there exists a partial isometry $U \in \mathcal{Q}$ such that $t = \operatorname{Ad}(U) \circ t'$ and $t' = \operatorname{Ad}(U^*) \circ t$. Now notice that U commutes with E_e. Thus for any $e \in G^1$ we have $\tau(s_e s_e^*) = \sum_{s(f)=r(e)} \tau(s_f s_f^*) = \sum_{s(f)=r(e)} t(s_f s_f^*) = t(s_e^* s_e)$ and

$$\begin{aligned}
d_{\tau,t}(e) - d_{\tau,t'}(e) &= -\operatorname{ind}_{E_e} \tau(s_e)t(s_e^*) + \operatorname{ind}_{E_e} \tau(s_e)t'(s_e^*) \\
&= \operatorname{ind}_{E_e} t(s_e)\tau(s_e^*) + \operatorname{ind}_{E_e} \tau(s_e)t'(s_e^*) \\
&= \operatorname{ind}_{E_e} t(s_e)\tau(s_e^* s_e)t'(s_e^*) \qquad \text{by Lemma 4.1.5} \\
&= \operatorname{ind}_{E_e} t(s_e)t'(s_e^*) \\
&= -d_{t,t'}(e).
\end{aligned}$$

Hence $d_{\tau,t} - d_{\tau,t'} = -d_{t,t'}$. Now let $k \in \prod_{G^1} \mathbb{Z}$ be the vector given by $k(f) := \operatorname{ind}_{E_f} U$. Then for any $e \in G^1$ we have

$$\begin{aligned}
d_{t,t'}(e) &= -\operatorname{ind}_{E_e} t(s_e)t'(s_e^*) \\
&= -\operatorname{ind}_{E_e} t(s_e)Ut(s_e^*)U^* \\
&= -\operatorname{ind}_{E_e} t(s_e)Ut(s_e^*) - \operatorname{ind}_{E_e} U^* \qquad \text{by Lemma 4.1.5} \\
&= -\operatorname{ind}_{t(s_e^* s_e)} U - \operatorname{ind}_{E_e} U^* \qquad \text{by Lemma 4.1.11} \\
&= -\operatorname{ind}_{\sum\limits_{s(f)=r(e)} E_f} U + \operatorname{ind}_{E_e} U \\
&= -\sum_{s(f)=r(e)} \operatorname{ind}_{E_f} U + \operatorname{ind}_{E_e} U \qquad \text{by Lemma 4.1.4}
\end{aligned}$$

$$= -\left(\sum_{f \in G^1} B_G(e,f)k(f) - k(e)\right)$$

so $d_{t,t'} = -(B_G - I)k$ and $d_{\tau,t} - d_{\tau,t'} = -d_{t,t'} \in \operatorname{im}(B_G - I)$. $\square$

Definition 4.1.13. Let G be a row-finite graph with no sinks that satisfies Condition (L). Let B_G be the edge matrix of G and view $B_G - I$ as an endomorphism on $\prod_{G^1} \mathbb{Z}$. If τ is an essential extension of $C^*(G)$, then we shall define an element $d_\tau \in \operatorname{coker}(B_G - I)$ by

$$d_\tau := [d_{\tau,t}] \in \operatorname{coker}(B_G - I),$$

where t is any degenerate extension with the property that $t(s_e s_e^*) = \tau(s_e s_e^*)$ for all $e \in G^1$.

In the above definition, the existence of t follows from Lemma 4.1.7. In addition, since any two degenerate essential extensions are CK-equivalent, it follows from Proposition 4.1.12 that the class of $d_{\tau,t}$ in $\operatorname{coker}(B_G - I)$ will be independent of the choice of t. Therefore d_τ is well-defined.

Lemma 4.1.14. *Suppose that τ_1 and τ_2 are extensions of a C^*-algebra A, and that v is a partial isometry in $\mathcal{Q}$ for which $\tau_1 = \operatorname{Ad}(v) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}(v^*) \circ \tau_1$. Then there exists either an isometry or coisometry $W \in \mathcal{B}$ such that $\tau_1 = \operatorname{Ad}\pi(W) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}\pi(W^*) \circ \tau_1$.*

Proof. Since v is a partial isometry Lemma 4.1.2 tells us that there exists a partial isometry $V \in \mathcal{B}$ such that $\pi(V) = v$. If we consider the projections $1 - V^*V$ and $1 - VV^*$, then one of these projections has a rank greater than or equal to the rank of the other.

Let us suppose first that the rank of $1 - VV^*$ is greater than or equal to the rank of $1 - V^*V$. Then we may choose a partial isometry V_0 in $\mathcal{B}$ with source projection

$V_0^*V_0 = 1 - V^*V$ and range projection $V_0V_0^* \leq 1 - VV^*$. If we define $W = V + V_0$, then W is an isometry. If we now let $w := \pi(W)$, then $wv^*v = \pi(WV^*V) = \pi((V + V_0)V^*V) = \pi(VV^*V) = \pi(V) = v$. Thus $\operatorname{Ad}(w) \circ \tau_2 = \operatorname{Ad}(w) \circ \operatorname{Ad}(v^*) \circ \tau_1 = \operatorname{Ad}(w) \circ \operatorname{Ad}(v^*) \circ \operatorname{Ad}(v) \circ \tau_2 = \operatorname{Ad}(wv^*v) \circ \tau_2 = \operatorname{Ad}(v) \circ \tau_2 = \tau_1$. A similar argument shows that $\operatorname{Ad}(w^*) \circ \tau_1 = \tau_2$.

On the other hand, if it is the case that the rank of $1 - VV^*$ is less than the rank of $1 - V^*V$, then we may choose a partial isometry V_0 in $\mathcal{B}$ with source projection $V_0^*V_0 = 1 - VV^*$ and range projection $V_0V_0^* \leq 1 - V^*V$. Then $W = V + V_0^*$ will be a coisometry, and a calculation similar to the one above shows that v may be replaced by $w = \pi(W)$. □

Corollary 4.1.15. *Let τ_1 and τ_2 be essential extensions of a C^*-algebra A. Then τ_1 and τ_2 are CK-equivalent if and only if there exists either an isometry or coisometry W in $\mathcal{B}$ such that $\tau_1 = \operatorname{Ad} \pi(W) \circ \tau_2$ and $\tau_2 = \operatorname{Ad} \pi(W^*) \circ \tau_1$.*

Lemma 4.1.16. *Let G be a row-finite graph with no sinks that satisfies Condition (L). Suppose that τ_1 and τ_2 are two essential extensions of $C^*(G)$ that are equal in $\operatorname{Ext}(C^*(G))$. Then d_{τ_1} and d_{τ_2} are equal in $\operatorname{coker}(B_G - I)$.*

Proof. Since τ_1 and τ_2 are equal in $\operatorname{Ext}(C^*(G))$ it follows that they are CK-equivalent. By interchanging τ_1 and τ_2 if necessary, we may use Corollary 4.1.15 to choose an isometry W in $\mathcal{B}$ for which $\tau_1 = \operatorname{Ad} \pi(W) \circ \tau_2$ and $\tau_2 = \operatorname{Ad} \pi(W^*) \circ \tau_1$. For each $e \in G^1$ define $E_e := \tau_1(s_e s_e^*)$ and $F_e := \tau_2(s_e s_e^*)$. By Lemma 4.1.7 there exists a degenerate essential extension $t_2 = \pi \circ \tilde{t}_2$ with the property that $t_2(s_e s_e^*) = \tau_2(s_e s_e^*) = F_e$ for all $e \in G^1$. Then $\tilde{t}_1 := W\tilde{t}_2W^*$ will be a representation of $C^*(G)$ ($\tilde{t}_1$ is multiplicative since W is an isometry), and thus $t_1 := \pi \circ \tilde{t}_1$ will be a degenerate extension with the property that $t_1(s_e s_e^*) = \tau_1(s_e s_e^*)$. Now since τ_1 is essential we have that

$$t_1(p_v) = \sum_{s(e)=v} t_1(s_e s_e^*) = \sum_{s(e)=v} \tau_1(s_e s_e^*) = \tau_1(p_v) \neq 0.$$

Therefore $p_v \notin \ker t_1$ for all $v \in G^0$ and it follows from Lemma 4.1.6 that $\ker t_1 = \{0\}$, and thus t_1 is essential.

Now recall that $E_e := \tau_1(s_e s_e^*)$ and $F_e := \tau_2(s_e s_e^*)$. Since W is an isometry, we see that $\pi(W)F_e$ is a partial isometry with source projection F_e and range projection E_e. Therefore by Lemma 4.1.11 it follows that

$$\begin{aligned}
\operatorname{ind}_{F_e} \tau_2(s_e)t_2(s_e^*) &= \operatorname{ind}_{E_e} \pi(W)F_e\tau_2(s_e)t_2(s_e^*)F_e\pi(W^*) \\
&= \operatorname{ind}_{E_e} \pi(W)\tau_2(s_e)t_2(s_e^*)\pi(W^*) \\
&= \operatorname{ind}_{E_e} \pi(W)\tau_2(s_e)\pi(W^*)\pi(W)t_2(s_e^*)\pi(W^*) \\
&= \operatorname{ind}_{E_e} \tau_1(s_e)t_1(s_e^*)
\end{aligned}$$

and d_{τ_2} equals d_{τ_1} in $\operatorname{coker}(B_G - I)$. □

Definition 4.1.17. If G is a row-finite graph with no sinks that satisfies Condition (L), we define the *Cuntz-Krieger map* to be the map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ defined by $\tau \mapsto d_\tau$.

The previous lemma shows that the Cuntz-Krieger map d is well-defined, and the next lemma shows that it is a homomorphism.

Lemma 4.1.18. *Suppose that G is a row-finite graph with no sinks that satisfies Condition (L). Then the Cuntz-Krieger map is additive.*

Proof. Let τ_1 and τ_2 be elements of $\operatorname{Ext}(C^*(G))$. Without loss of generality we may assume that $\tau_1 \perp \tau_2$. Let t_1 and t_2 be degenerate essential extensions such that

$t_1(s_e s_e^*) = \tau_1(s_e s_e^*)$ and $t_2(s_e s_e^*) = \tau_2(s_e s_e^*)$.

Because $\tau_1 \perp \tau_2$ we know that there exist orthogonal projections $p_1, p_2 \in \mathcal{Q}$ such that $\tau_i(C^*(G)) \subseteq p_i \mathcal{Q} p_i$ for $i \in \{1,2\}$. Now for any $e \in E^1$ and $i \in \{1,2\}$ we have that

$$\begin{aligned} p_i t_i(s_e) p_i =& p_i t_i(s_e s_e^*) t_i(s_e) t_i(s_e^* s_e) p_i = p_i \tau_i(s_e s_e^*) t_i(s_e) \tau_i(s_e^* s_e) p_i \\ =& \tau_i(s_e s_e^*) t_i(s_e) \tau_i(s_e^* s_e) = t_i(s_e s_e^*) t_i(s_e) t_i(s_e^* s_e) = t_i(s_e) \end{aligned}$$

Thus $t_i(s_e) \in p_i \mathcal{Q} p_i$ for all $e \in E^1$. Since the s_e's generate $C^*(G)$ it follows that $t_i(C^*(G)) \subseteq p_i \mathcal{Q} p_i$. Thus $t_1 \perp t_2$, and we may form the essential extension $t_1 \boxplus t_2$ given by $a \mapsto t_1(a) + t_2(a)$.

Notice that t_1 and t_2 are degenerate extensions, and thus $t_1 + t_2$ is a degenerate extension. Because $t := t_1 \boxplus t_2$ is weakly stably equivalent to $t_1 + t_2$ we see that t is in the zero class in Ext. But since t is an essential extension with the property that $t(s_e s_e^*) = \tau_1(s_e s_e^*) + \tau_2(s_e s_e^*)$, it follows from Lemma 4.1.12 that $d_{\tau \boxplus \tau_2} = [d_{\tau_1 \boxplus \tau_2, t}]$ in $\operatorname{coker}(B_G - I)$. Furthermore, since $\operatorname{ind}_E X = \operatorname{ind}_E EX = \operatorname{ind}_E XE$, we have that

$$\begin{aligned} d_{\tau_1 \boxplus \tau_2, t}(e) =& -\operatorname{ind}_{(\tau_1 \boxplus \tau_2)(s_e s_e^*)} (\tau_1 \boxplus \tau_2)(s_e) t(s_e^*) \\ =& -\operatorname{ind}_{\tau_1(s_e s_e^*)} (\tau_1 \boxplus \tau_2)(s_e) t(s_e^*) \\ & \qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} (\tau_1 \boxplus \tau_2)(s_e) t(s_e^*) \qquad \text{(by Lemma 4.1.4)} \\ =& -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e s_e^*)(\tau_1(s_e) + \tau_2(s_e)) t(s_e^*) \\ & \qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e s_e^*)(\tau_1(s_e) + \tau_2(s_e)) t(s_e^*) \\ =& -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e) t(s_e^*) - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e) t(s_e^*) \\ =& -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e)(t_1(s_e^*) + t_2(s_e^*)) \tau_1(s_e s_e^*) \\ & \qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e)(t_1(s_e^*) + t_2(s_e^*)) \tau_2(s_e s_e^*) \end{aligned}$$

$$
\begin{aligned}
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e)(t_1(s_e^*) + t_2(s_e^*))t_1(s_e s_e^*) \\
&\qquad\qquad\qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e)(t_1(s_e^*) + t_2(s_e^*))t_2(s_e s_e^*) \\
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e)t_1(s_e^*) - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e)t_2(s_e^*) \\
&= d_{\tau_1,t_1}(e) + d_{\tau_2,t_2}(e).
\end{aligned}
$$

So $d_{\tau_1 \boxplus \tau_2, t} = d_{\tau_1,t} + d_{\tau_2,t}$. Also since $\tau_1 \boxplus \tau_2$ is weakly stably equivalent to $\tau_1 + \tau_2$, Lemma 4.1.16 implies that we have $d_{\tau_1 \boxplus \tau_2} = d_{\tau_1+\tau_2}$ in $\operatorname{coker}(B_G - I)$. Putting this all together gives $d_{\tau_1+\tau_2} = d_{\tau_1 \boxplus \tau_2} = [d_{\tau_1 \boxplus \tau_2,t}] = [d_{\tau_1,t_1} + d_{\tau_2,t_2}] = [d_{\tau_1,t_1}] + [d_{\tau_2,t_2}] = d_{\tau_1} + d_{\tau_2}$ in $\operatorname{coker}(B_G - I)$. Thus d is additive. □

We mention the following two lemmas, both of whose proofs are straightforward.

Lemma 4.1.19. *Let $E \in \mathcal{Q}$ be a projection, and suppose that T is a unitary in $E\mathcal{Q}E$ with $\operatorname{ind}_E T = 0$. If $E' \in \mathcal{B}$ is a projection such that $\pi(E') = E$, then there is a unitary $U \in \mathcal{B}(E'\mathcal{H})$ such that $\pi(U) = T$.*

Lemma 4.1.20. *Suppose that $\mathcal{H}_1, \mathcal{H}_2, \ldots$ is a countable collection of pairwise orthogonal subspaces of the Hilbert space $\mathcal{H}$, and for each $i \in \{1, 2, \ldots\}$ V_i is an operator in $\mathcal{B}(\mathcal{H}_i)$ with norm 1. If we extend each V_i to all of $\mathcal{H}$ by defining it to be zero on $\mathcal{H}_i^\perp$, then the sum $\sum_{i=1}^\infty V_i$ converges in the strong operator topology on $\mathcal{B}(\mathcal{H})$ to an operator of norm 1.*

Proposition 4.1.21. *Let G be a row-finite graph with no sinks that satisfies Condition (L). Then the Cuntz-Krieger map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ defined by $\tau \mapsto d_\tau$ is injective.*

Proof. Let τ be an essential extension of $C^*(G)$ and suppose that d_τ equals 0 in $\operatorname{coker}(B_G - I)$. Use Lemma 4.1.7 to choose a degenerate essential extension $t := \pi \circ \tilde{t}$ of $C^*(G)$ such that $t(s_e s_e^*) = E_e := \tau(s_e s_e^*)$ for all $e \in G^1$. Also let $E'_e := \tilde{t}(s_e s_e^*)$.

By hypothesis, there exists $k \in \prod_{G^1} \mathbb{Z}$ such that $d_{\tau,t} = (B_G - I)k$. Since τ is essential, for all $e \in G^1$ we must have that $\pi(E'_e) = E_e = \tau(s_e s_e^*) \neq 0$. Since E'_e is a projection, this implies that $\dim(\operatorname{im}(E'_e)) = \infty$. Therefore for each $e \in G^1$ we may choose isometries or coisometries V_e in $\mathcal{B}(E'_e(\mathcal{H}))$ such that $\operatorname{ind}_{E_e} V_e = -k(e)$. Extend each V_e to all of $\mathcal{H}$ by defining it to be zero on $(E'_e(\mathcal{H}))^\perp$. Let $U := \sum_{e \in G^1} V_e$. It follows from Lemma 4.1.20 that this sum converges in the strong operator topology. Notice that for all $e, f \in G^1$ we have

$$V_f \tilde{t}(s_e s_e^*) = V_f E'_f E'_e = \begin{cases} V_f & \text{if } e = f \\ 0 & \text{otherwise.} \end{cases}$$

Since U commutes with E'_e for all $e \in G^1$, we see that $\pi(U)\tau(s_e)\pi(U^*)t(s_e^*)$ is a unitary in $E_e \mathcal{Q} E_e$. Hence we may consider $\operatorname{ind}_{E_e} \pi(U)\tau(s_e)\pi(U^*)t(s_e^*)$. Using the above identity we see that for each $e \in G^1$ we have

$$\begin{aligned}
\operatorname{ind}_{E_e} \pi(U)\tau(s_e)\pi(U^*)t(s_e^*) &= \operatorname{ind}_{E_e} \pi(U)\tau(s_e s_e^*)\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e} \pi(U\tilde{t}(s_e s_e^*))\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e} \pi\left(\sum_{f \in G^1} V_f \tilde{t}(s_e s_e^*)\right)\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e} \pi(V_e \tilde{t}(s_e s_e^*))\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e} \pi(V_e)\tau(s_e)t(s_e^*)\Big(t(s_e)\pi(U^*)t(s_e^*)\Big). \qquad (4.3)
\end{aligned}$$

Now since $t(s_e)$ is a partial isometry with source projection

$$t(s_e^* s_e) = \sum_{s(f)=r(e)} t(s_f s_f^*) = \sum_{s(f)=r(e)} E_f$$

and range projection $t(s_e s_e^*) = E_e$, we may use Lemma 4.1.11 to conclude that

$$\operatorname{ind}_{\sum\limits_{s(f)=r(e)} E_f} \pi(U^*) = \operatorname{ind}_{E_e} t(s_e)\pi(U^*)t(s_e^*).$$

This combined with Lemma 4.1.4 implies that

$$\begin{aligned}
\operatorname{ind}_{E_e} t(s_e)\pi(U^*)t(s_e^*) &= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} \pi(U^*) \\
&= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} E_f \pi\left(\sum_{g\in G^1} V_g^*\right) E_f \\
&= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} \pi\left(\sum_{g\in G^1} E_f' V_g^* E_f'\right) \\
&= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} \pi(V_f^*) \\
&= \sum_{s(f)=r(e)} k(f) \\
&= \sum_{f\in G^1} B_G(e,f)k(f). \qquad (4.4)
\end{aligned}$$

Combining (4.3) and (4.4) with Lemma 4.1.5 gives

$$\operatorname{ind}_{E_e} \pi(U)\tau(s_e)\pi(U^*)t(s_e^*) = \left(\sum_{f\in G^1} B_G(e,f)k(f) - k(e)\right) - d_\tau(e) = 0.$$

Thus by Lemma 4.1.19 there exists an operator $X_e \in \mathcal{B}$ such that the restriction of X_e to $E_e'(\mathcal{H})$ is a unitary operator and $\pi(X_e) = \pi(U)\tau(s_e)\pi(U^*)t(s_e^*)$. Let $T_e := X_e\tilde{t}(s_e)$. Then T_e is a partial isometry that satisfies $T_eT_e^* = E_e'$ and $T_e^*T_e = \tilde{t}(s_e^*)X_e^*X_e\tilde{t}(s_e) = \tilde{t}(s_e^*s_e) = \tilde{t}(p_{r(e)})$. One can then check that $\{\tilde{t}(p_v), T_e\}$ is a Cuntz-Krieger G-family in $\mathcal{B}$. Thus by the universal property of $C^*(G)$ there exists a homomorphism $\tilde{\rho} : C^*(G) \to \mathcal{B}$ such that $\tilde{\rho}(p_v) = \tilde{t}(p_v)$ and $\tilde{\rho}(s_e) = T_e$. Let $\rho := \pi \circ \tilde{\rho}$. Then ρ is a degenerate extension of $C^*(G)$. Furthermore, since $\rho(p_v) = t(p_v) \neq 0$ we see that

$p_v \notin \ker \rho$ for all $v \in G^0$. Since G satisfies Condition (L), it follows from Lemma 4.1.6 that $\ker \rho = \{0\}$ and ρ is a degenerate essential extension. In addition, we see that for each $e \in G^1$

$$\begin{aligned}
\rho(s_e) &= \pi(T_e) \\
&= \pi(X_e \tilde{t}(s_e)) \\
&= \pi(U)\tau(s_e)\pi(U^*)t(s_e^*)t(s_e) \\
&= \pi(U)\tau(s_e)\pi\Big(U^* \sum_{s(g)=r(e)} \tilde{t}(s_g s_g^*)\Big) \\
&= \pi(U)\tau(s_e)\pi\Big(\sum_{f \in G^1} V_f^* \sum_{s(g)=r(e)} E'_g\Big) \\
&= \pi(U)\tau(s_e)\pi\Big(\sum_{s(g)=r(e)} E'_g \sum_{f \in G^1} V_f^*\Big) \\
&= \pi(U)\tau(s_e)\tau(s_e^* s_e)\pi\Big(\sum_{f \in G^1} V_f^*\Big) \\
&= \pi(U)\tau(s_e)\pi(U^*).
\end{aligned}$$

Thus $\rho(s_e) = \pi(U)\tau(s_e)\pi(U^*)$ for all $e \in G^1$, and since the s_e's generate $C^*(G)$, it follows that $\rho(a) = \pi(U)\tau(a)\pi(U^*)$ for all $a \in C^*(G)$ and hence $\rho = \mathrm{Ad}(\pi(U)) \circ \tau$.

In addition, since the V_e's are either isometries or coisometries on $E'_e(\mathcal{H})$ with finite Fredholm index, it follows that $\pi(V_e^* V_e) = \pi(V_e V_e^*) = \pi(E'_e)$. Therefore, for any $e \in G^1$ we have that

$$\begin{aligned}
\pi(U^*U)\tau(s_e) &= \pi\left(U^* \sum_{f \in G^1} V_f \tilde{t}(s_e s_e^*)\right)\tau(s_e) \\
&= \pi\left(U^* V_e E'_e\right)\tau(s_e) \\
&= \pi\left(\sum_{f \in G^1} V_f^* E'_e V_e\right)\tau(s_e)
\end{aligned}$$

$$
\begin{aligned}
&= \pi(V_e^* V_e)\tau(s_e) \\
&= \pi(E'_e)\tau(s_e) \\
&= \tau(s_e s_e^*)\tau(s_e) \\
&= \tau(s_e).
\end{aligned}
$$

Again, since the s_e's generate $C^*(G)$, it follows that $\pi(U^*U)\tau(a) = \tau(a)$ for all $a \in C^*(G)$. Similarly, $\tau(a)\pi(U^*U) = \tau(a)$ for all $a \in C^*(G)$. Thus $\pi(U^*)\rho(a)\pi(U) = \pi(U^*U)\tau(a)\pi(U^*U) = \tau(a)$ for all $a \in C^*(G)$ and $\tau = \mathrm{Ad}(\pi(U)^*) \circ \rho$.

Now because the V_e's are all isometries or coisometries on orthogonal spaces, it follows that U, and hence $\pi(U)$, is a partial isometry. Therefore, $\tau = \rho$ in $\mathrm{Ext}(C^*(G))$ and since ρ is a degenerate essential extension it follows that $\tau = 0$ in $\mathrm{Ext}(C^*(G))$. This implies that d is injective. □

4.2 The Wojciech map

In the previous section we showed that if G is a row-finite graph with no sinks that satisfies Condition (L), then the Cuntz-Krieger map $d : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(B_G - I)$ is a monomorphism. It turns out that d is also surjective; that is, it is an isomorphism. In this section we shall prove this fact, but we shall do it in an indirect way. We show that $\mathrm{coker}(B_G - I)$ is isomorphic to $\mathrm{coker}(A_G - I)$ and then compose d with this isomorphism to get a map from $\mathrm{Ext}(C^*(G))$ into $\mathrm{coker}(A_G - I)$. We call this composition the Wojciech map and we shall show that it, and consequently also d, is surjective. For the rest of this chapter we will be mostly concerned with the Wojciech map and how it relates to 1-sink extensions defined in Chapter 2.

Definition 4.2.1. Let G be a graph. The *source matrix* of G is the $G^0 \times G^1$ matrix given by

$$S_G(v,e) = \begin{cases} 1 & \text{if } s(e) = v \\ 0 & \text{otherwise} \end{cases}$$

and the *range matrix* of G is the $G^1 \times G^0$ matrix given by

$$R_G(e,v) = \begin{cases} 1 & \text{if } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if G is a row-finite graph, then S_G will have rows that are eventually zero and left multiplication by S_G defines a map $S_G : \prod_{G^1} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$. Also R_G will always have rows that are eventually zero. (In fact, regardless of any conditions on G, R_G will have only one nonzero entry in each row.) Therefore left multiplication by R_G defines a map $R_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^1} \mathbb{Z}$. Furthermore, one can see that

$$R_G S_G = B_G \quad \text{and} \quad S_G R_G = A_G.$$

The following lemma is well known for finite graphs and a proof for S_G restricted to the direct sum $S_G : \bigoplus_{G^1} \mathbb{Z} \to \bigoplus_{G^0} \mathbb{Z}$ is given in [59, Lemma 4.2]. Essentially the same proof goes through if we replace the direct sums by direct products.

Lemma 4.2.2. *Let G be a row-finite graph. The map $S_G : \prod_{G^1} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ induces an isomorphism $\overline{S_G} : \operatorname{coker}(B_G - I) \to \operatorname{coker}(A_G - I)$.*

Proof. Suppose that $z \in \operatorname{im}(B_G - I)$. Then $z = (B_G - I)u$ for some $u \in \prod_{G^1} \mathbb{Z}$. Then

$$S_G z = S_G(B_G - I)u = S_G(R_G S_G - I)u = (S_G R_G - I)S_G u = (A_G - I)S_G u$$

and S_G does in fact map $\operatorname{im}(B_G - I)$ into $\operatorname{im}(A_G - I)$. Thus S_G induces a homomorphism $\overline{S_G}$ of $\operatorname{coker}(B_G - I)$ into $\operatorname{coker}(A_G - I)$. In the same way, R_G induces a homomorphism $\overline{R_G}$ from $\operatorname{coker}(A_G - I)$ into $\operatorname{coker}(B_G - I)$, which we claim is an inverse for $\overline{S_G}$. We see that

$$\begin{aligned}\overline{R_G} \circ \overline{S_G}(u + \operatorname{im}(B_G - I)) &= R_G S_G u + \operatorname{im}(B_G - I) \\ &= u + (B_G u - u) + \operatorname{im}(B_G - I) \\ &= u + \operatorname{im}(B_G - I)\end{aligned}$$

and similarly $\overline{S_G} \circ \overline{R_G}$ is the identity on $\operatorname{coker}(A_G - I)$. □

Definition 4.2.3. Let G be a row-finite graph with no sinks that satisfies Condition (L), and let $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ be the Cuntz-Krieger map. The *Wojciech map* is the homomorphism $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ given by $\omega := \overline{S_G} \circ d$. Given an extension τ of $C^*(G)$, we shall refer to the class $\omega(\tau)$ in $\operatorname{coker}(A_G - I)$ as the *Wojciech class* of τ.

Lemma 4.2.4. *Let G be a row-finite graph with no sinks that satisfies Condition (L). Then the Wojciech map associated to G is a monomorphism.*

Proof. Since $\omega = \overline{S_G} \circ d$, and $\overline{S_G}$ is an isomorphism by Lemma 4.2.2, the result follows from Proposition 4.1.21. □

We shall eventually show that the Wojciech map is also surjective; that is, it is an isomorphism. In order to do this we consider 1-sink extensions, which were introduced in Definition 2.1.1 of Chapter 2, and describe a way to associate elements of $\operatorname{Ext}(C^*(G))$ to them.

If (E, v_0) is a 1-sink extension of G, then we may let $\pi_E : C^*(E) \to C^*(G)$ be the surjection described in Corollary 2.1.3. Then $\ker \pi_E = I_{v_0}$ where I_{v_0} is the ideal in

$C^*(E)$ generated by the projection p_{v_0}. Thus we have a short exact sequence

$$0 \longrightarrow I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \longrightarrow 0.$$

We call E an *essential* 1-sink extension of G when $G^0 \geq v_0$, and from Lemma 2.2.2 I_{v_0} is an essential ideal of $C^*(E)$ if and only if E is an essential 1-sink extension of G.

Lemma 4.2.5. *If G is a row-finite graph and (E, v_0) is an essential 1-sink extension of G, then $I_{v_0} \cong \mathcal{K}$.*

Proof. Let $E^*(v_0)$ be the set of all paths in E whose range is v_0. Since E is an essential 1-sink extension of G, it follows that $G^0 \geq v_0$. Thus for every $w \in G^0$ there exists a path from w to v_0. If G^0 is infinite, this implies that $E^*(v_0)$ is also infinite. If G^0 is finite, then because $G^0 \geq v_0$ it follows that G is a finite graph with no sinks, and hence contains a loop. If w is any vertex on this loop, then there is a path from w to v_0 and hence $E^*(v_0)$ is infinite. Now because $E^*(v_0)$ is infinite it follows from [54, Corollary 2.2] that $I_{v_0} \cong \mathcal{K}(\ell^2(E^*(v_0))) \cong \mathcal{K}$. $\square$

Definition 4.2.6. Let G be a row-finite graph and let (E, v_0) be an essential 1-sink extension of G. *The extension associated to E* is (the strong equivalence class of) the Busby invariant of any extension

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_E} C^*(E) \xrightarrow{\pi_E} C^*(G) \longrightarrow 0$$

where i_E is any isomorphism from $\mathcal{K}$ onto I_{v_0}. As with other extensions we shall not distinguish between an extension and its Busby invariant.

Remark 4.2.7. The above extension is well-defined up to strong equivalence. If different choices of i_E are made then it follows from a quick diagram chase that the two associated extensions will be strongly equivalent (see problem 3E(c) of [102] for more details). Also recall that since p_{v_0} is a minimal projection in I_{v_0} [54, Corollary 2.2], it follows that $i_E^{-1}(p_{v_0})$ will always be a rank 1 projection in $\mathcal{K}$.

Let (E, v_0) be a 1-sink extension of G. Then for $w \in E^0$ we denote by $Z(w, v_0)$ the set of paths α from w to v_0 with the property that α immediately leaves G; that is, $\alpha_i \in E^1 \backslash G^1$ for $1 \leq i \leq |\alpha|$. Recall that the *Wojciech vector* of E is the element $\omega_E \in \prod_{G^0} \mathbb{N}$ given by

$$\omega_E(w) := \#Z(w, v_0).$$

Also recall that an edge $e \in E^1$ with $s(e) \in G^0$ and $r(e) \notin G^0$ is called a *boundary edge*, and the sources of these edges are called *boundary vertices*.

Lemma 4.2.8. *Let G be a row-finite graph and let (E, v_0) be a 1-sink extension of G. If $\{s_e, p_v\}$ is the canonical Cuntz-Krieger E-family in $C^*(E)$ and $\sigma : C^*(E) \to \mathcal{B}$ is a representation with the property that $\sigma(p_{v_0})$ is a rank 1 projection, then*

$$\operatorname{rank} \sigma(s_e) = \#Z(r(e), v_0) \qquad \textit{for all } e \in E^1 \backslash G^1.$$

Proof. For $e \in E^1 \backslash G^1$ let $k_e := \max\{|\alpha| : \alpha \in Z((r(e), v_0)\}$. Since E is a 1-sink extension of G we know that k_e is finite. We shall prove the claim by induction on k_e. If $k_e = 0$, then $r(e) = v_0$ and $\operatorname{rank} \sigma(s_e) = \operatorname{rank} \sigma(s_e^* s_e) = \operatorname{rank} \sigma(p_{v_0}) = 1$.

Assume that the claim holds for all $f \in E^1 \backslash G^1$ with $k_f \leq m$. Then let $e \in E^1 \backslash G^1$ with $k_e = m + 1$. Since E is a 1-sink extension of G there are no loops based at $r(e)$. Thus $k_f \leq m$ for all $f \in E^1 \backslash G^1$ with $s(f) = r(e)$. By the induction hypothesis $\operatorname{rank} \sigma(s_f) = \#Z(r(e), v_0)$ for all f with $s(f) = r(e)$. Since the projections $s_f s_f^*$ are mutually orthogonal we have

$$\begin{aligned}\operatorname{rank}\sigma(s_e) = \operatorname{rank}\sigma(s_e^*s_e) = \operatorname{rank}\left(\sum_{s(f)=r(e)}\sigma(s_fs_f^*)\right) = \sum_{s(f)=r(e)}\operatorname{rank}\sigma(s_fs_f^*)\\ = \sum_{s(f)=r(e)}\#Z((r(f),v_0) = \#Z(r(e),v_0).\end{aligned}$$

□

Lemma 4.2.9. *Let G be a row-finite graph with no sinks that satisfies Condition (L), and let $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ be the Cuntz-Krieger map. If (E, v_0) is an essential 1-sink extension of G and τ is the Busby invariant of the extension associated to E, then*

$$d(\tau) = [x]$$

where $[x]$ is the class in $\operatorname{coker}(B_G - I)$ of the vector $x \in \prod_{G^1}\mathbb{Z}$ given by $x(e) := \omega_E(r(e))$ for all $e \in G^1$, and ω_E is the Wojciech vector of E.

Proof. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger G-family in $C^*(G)$, and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. Choose an isomorphism $i_E : \mathcal{K} \to I_{v_0}$, and let σ and τ be the homomorphisms that make the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \xrightarrow{i_E} & C^*(E) & \xrightarrow{\pi_E} & C^*(G) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

commute (see [102, §3.2] for details). Then τ is the Busby invariant of the extension associated to E, and since E is an essential 1-sink extension, it follows that σ and τ are injective. For all $v \in E^0$ and $e \in E^1$ define

$$H_v := \operatorname{im}\sigma(q_v) \qquad \text{and} \qquad H_e := \operatorname{im}\sigma(t_et_e^*).$$

Note that $s(e) = v$ implies that $H_e \subseteq H_v$. Also since $i_E^{-1}(q_{v_0})$ is a rank 1 projection, and since the above diagram commutes, it follows that $\sigma(q_{v_0})$ is a rank 1 projection. Thus H_{v_0} is 1-dimensional. Furthermore, by Lemma 4.2.8 we see that $\dim(H_v) = \#Z(v, v_0)$ and $\dim(H_e) = \#Z(r(e), v_0)$ for all $v \in E^0 \backslash G^0$ and $e \in E^1 \backslash G^1$. In addition, since $t_e t_e^* \leq q_{s(e)}$ for any $e \in E^1 \backslash G^1$ and because the q_v's are mutually orthogonal projections, it follows that the H_e's are mutually orthogonal subspaces for all $e \in E^1 \backslash G^1$.

For all $v \in G^0$ define

$$V_v := H_v \ominus \Big(\bigoplus_{\substack{e \text{ is a boundary} \\ \text{edge and } s(e)=v}} H_e \Big).$$

Then for every $v \in G^0$, we have $\pi(\sigma(q_v)) = \tau(\pi_E(q_v)) = \tau(p_v) \neq 0$ since τ is injective. Therefore, the rank of $\sigma(q_v)$ is infinite and hence $\dim(H_v) = \infty$ and $\dim(V_v) = \infty$. Now for each $v \in G^0$ and $e \in G^1$ let P_v be the projection onto V_v and S_e be a partial isometry with initial space $V_{r(e)}$ and final space H_e. One can then check that $\{S_e, P_v\}$ is a Cuntz-Krieger G-family in $\mathcal{B}$. Therefore, by the universal property of $C^*(G)$ there exists a homomorphism $\tilde{t} : C^*(G) \to \mathcal{B}$ with the property that $\tilde{t}(s_e) = S_e$ and $\tilde{t}(p_v) = P_v$. Define $t := \pi \circ \tilde{t}$.

Then for all $v \in G^0$ we have that

$$t(p_v) = \pi(\tilde{t}(p_v)) = \pi(P_v) \neq 0.$$

Thus $p_v \notin \ker t$ for all $v \in G^0$. By Lemma 4.1.6 it follows that $\ker t = \{0\}$ and t is an essential extension of $C^*(G)$. Now since $S_e S_e^*$ is a projection onto a subspace of $\operatorname{im} \sigma(t_e t_e^*)$ with finite codimension, it follows that $\pi(S_e S_e^*) = \pi(\sigma(t_e t_e^*))$. Thus t has the property that for all $e \in G^1$

$$t(s_e s_e^*) = \pi(\tilde{t}(s_e s_e^*)) = \pi(S_e S_e^*) = \pi(\sigma(t_e t_e^*)) = \tau(\pi_E(t_e t_e^*)) = \tau(s_e s_e^*).$$

By the definition of the Cuntz-Krieger map d it follows that the image of the extension associated to E will be the class of the vector d_τ in $\operatorname{coker}(B_G - I)$, where $d_\tau(e) = -\operatorname{ind}_{\tau(s_e s_e^*)} \tau(s_e)t(s_e^*)$. Now $\operatorname{ind}_{\tau(s_e s_e^*)} \tau(s_e)t(s_e^*)$ is equal to the Fredholm index of $\sigma(t_e t_e^*)\sigma(t_e)S_e^*\sigma(t_e t_e^*) = \sigma(t_e)S_e^*$ in $\operatorname{im}(\sigma(t_e t_e^*)) = H_e$. Since S_e is a partial isometry with initial space $V_{r(e)} \subseteq H_{r(e)}$ and final space H_e, and since $\sigma(t_e)$ is a partial isometry with initial space $H_{r(e)}$ it follows that $\ker \sigma(t_e)S_e^* = \{0\}$ in H_e. Furthermore, $\sigma(t_e^*)$ is a partial isometry with initial space H_e and final space

$$H_{r(e)} = V_{r(e)} \oplus \Big(\bigoplus_{\substack{f \text{ is a boundary} \\ \text{edge and } s(f)=r(e)}} H_f \Big)$$

and S_e is a partial isometry with initial space $V_{r(e)}$. Therefore, since $\dim(H_f) = \#Z(r(f), v_0)$ for all $f \notin G^1$ we have that

$$\ker((\sigma(t_e)S_e)^*) = \ker(S_e\sigma(t_e^*)) = \sum_{s(f)=r(e)} Z(r(f), v_0) = \omega_E(r(e)).$$

Thus $d_\tau(e) = \omega_E(r(e))$ for all $e \in G^1$. $\square$

Proposition 4.2.10. *Let G be a row-finite graph with no sinks that satisfies Condition (L), and suppose that (E, v_0) is an essential 1-sink extension of G. If τ is the Busby invariant of the extension associated to E, then the value that the Wojciech map $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ assigns to τ is given by the class of the*

Wojciech vector in $\operatorname{coker}(A_G - I)$*; that is,*

$$\omega(\tau) = [\omega_E].$$

Proof. From Lemma 4.2.9 we have that $d_\tau = [x]$ in $\operatorname{coker}(B_G - I)$, where $x \in \prod_{G^1} \mathbb{Z}$ is the vector given by $x(e) := \omega_E(r(e))$ for $e \in G^1$. By the definition of ω we have that $\omega(\tau) := \overline{S_G}(d_\tau)$ in $\operatorname{coker}(A_G - I)$. Thus $\omega(\tau)$ equals the class of the vector $y \in \prod_{G^0} \mathbb{Z}$ given by

$$y(v) = (S_G(x))(v) = \sum_{s(e)=v} x(e) = \sum_{s(e)=v} \omega_E(r(e)).$$

Hence for all $v \in G^0$ we have

$$y(v) - \omega_E(v) = \sum_{s(e)=v} \omega_E(r(e)) - \omega_E(v) = \sum_{w \in G^0} A_G(v,w)\omega_E(w) - \omega_E(v)$$

so $y - \omega_E = (A_G - I)\omega_E$. Thus $[y] = [\omega_E]$ and $\omega(\tau) = [\omega_E]$ in $\operatorname{coker}(A_G - I)$. $\square$

This result gives us a method to prove that ω is surjective. We need only produce essential 1-sink extensions with the appropriate Wojciech vectors.

Recall that a 1-sink extension E of G is said to be *simple* if $E^0 \backslash G^0$ consists of a single vertex. If G is a graph with no sinks, then for any $x \in \prod_{G^0} \mathbb{N}$ we may form a simple 1-sink extension of G with Wojciech vector equal to x merely by defining $E^0 := G^0 \cup \{v_0\}$ and $E^1 := G^1 \cup \{e_w^i : w \in G^0 \text{ and } 1 \leq i \leq x(w)\}$ where each e_w^i is an edge with source w and range v_0. In order to show that the Wojciech map is surjective we will not only need to produce such 1-sink extensions, but also ensure that they are essential.

Lemma 4.2.11. *Let G be a row-finite graph with no sinks that satisfies Condition (L). There exists a vector $n \in \prod_{G^0} \mathbb{Z}$ with the following two properties:*

1. $(A_G - I)n \in \prod_{G^0} \mathbb{N}$

2. for all $v \in G^0$ *there exists* $w \in G^0$ *such that* $v \geq w$ *and* $((A_G - I)n)(w) \geq 1$.

Proof. Let $L \subseteq G^0$ be those vertices of G that feed into a loop; that is,

$$L := \{v \in G^0 : \text{there exists a loop } x \text{ in } G \text{ for which } v \geq r(x_1)\}.$$

Now consider the set $M := G^0 \backslash L$. Because G has no sinks, and because $v \in M$ and $v \geq w$ implies that $w \in M$, it follows that M cannot have a finite number of elements. Thus M is either empty or countably infinite. If $M \neq \emptyset$ then list the elements of M as $M = \{w_1, w_2, \ldots\}$. Now let $v_1^1 := w_1$. Choose an edge $e_1^1 \in G^1$ with the property that $s(e_1^1) = v_1^1$ and define $v_2^1 := r(e_1^1)$. Continue in this fashion: given v_k^1 choose an edge e_k^1 with $s(e_k^1) = v_k^1$ and define $v_{k+1}^1 := r(e_k^1)$. Then $v_1^1, v_2^1, \ldots$ are the vertices of an infinite path which are all elements of M. Since these vertices do not feed into a loop it follows that they are distinct; i.e. $v_i^1 \neq v_j^1$ when $i \neq j$.

Now if every element $w \in M$ has the property that $w \geq v_i^1$ for some i, then we shall stop. If not, choose the smallest $j \in \mathbb{N}$ for which $w_j \ngeq v_i^1$ for all $i \in \mathbb{N}$. Then define $v_1^2 := w_j$ and choose an edge e_1^2 with $s(e_1^2) = v_1^2$. Define $v_2^2 := r(e_1^2)$. Continue in this fashion: given v_k^2 choose an edge e_k^2 with $s(e_k^2) = v_k^2$ and define $v_{k+1}^2 := r(e_k^2)$. Then we produce a set of distinct vertices $v_1^2, v_2^2, v_3^2, \ldots$ that lie on the infinite path $e_1^2 e_2^2 e_3^2 \ldots$. Moreover, since $v_1^2 \ngeq v_i^1$ for all i we must have that the v_i^2's are also distinct from the v_i^1's.

Continue in this manner. Having produced an infinite path $e_1^k e_2^k e_3^k \ldots$ with distinct vertices $v_1^k, v_2^k, \ldots$ we stop if every element $w \in M$ has the property that $w \geq v_i^j$ for some $1 \leq i < \infty, 1 \leq j \leq k$. Otherwise, we choose the smallest $l \in \mathbb{N}$ such that $w_l \ngeq v_i^j$ for all $1 \leq i < \infty, 1 \leq j \leq k$. We define $v_1^{k+1} := w_l$. Given v_j^{k+1} we choose an edge e_j^{k+1} with $s(e_j^{k+1}) = v_j^{k+1}$. We then define $v_{j+1}^{k+1} := r(e_j^{k+1})$. Thus we produce an infinite path $e_1^{k+1}e_2^{k+1}\ldots$ with distinct vertices $v_1^{k+1}, v_2^{k+1}, \ldots$. Moreover, since $v_1^{k+1} \not\geq v_i^j$ for all $1 \leq i < \infty, 1 \leq j \leq k$, it follows that the v_i^{k+1}'s are distinct from the v_i^j's for $j \leq k$.

By continuing this process we are able to produce the following. For some $n \in \mathbb{N} \cup \{\infty\}$ there is a set of distinct vertices $S \subseteq M$ given by

$$S = \{v_j^k : 1 \leq j < \infty, 1 \leq k < n\}$$

with the property that $M \geq S$, and for any $v_j^k \in S$ there exists an edge $e_j^k \in G^1$ for which $s(e_j^k) = v_j^k$ and $r(e_j^k) = v_{j+1}^k$.

Now define

$$a_v = \begin{cases} 1 & \text{if } v \in L \\ j & \text{if } v = v_j^k \in S \\ 0 & \text{otherwise.} \end{cases}$$

and let $n := (a_v) \in \prod_{G^0} \mathbb{Z}$. We shall now show that n has the appropriate properties. We shall first show that $(A_G - I)n \in \prod_{G^0} \mathbb{N}$. Let $v \in G^0$ and consider four cases. (Throughout the following remember that the entries of n are nonnegative integers.)

Case 1: $A_G(v,v) \geq 1$. Then $((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) \geq 0$.

Case 2: $A_G(v,v) = 0, v \in L$. Since $A_G(v,v) = 0$ and v feeds into a loop, there must exist an edge $e \in G^1$ with $s(e) = v$ and $r(e) \in L$. Thus

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{r(e)}A_G(v, r(e)) \geq 1(-1) + 1(1) = 0.$$

Case 3: $A_G(v,v) = 0, v = v_j^k \in S$. Then there exists an edge e_j^k with $s(e_j^k) = v_j^k$ and

$r(e_j^k) = v_{j+1}^k \neq v_j^k$. Thus

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{v_{j+1}^k} A_G(v, v_{j+1}^k) \geq j(-1) + (j+1)(1) = 1.$$

Case 4: $A_G(v,v) = 0, v \notin L, v \notin S$. Then

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) \geq 0 \cdot (A_G(v,v) - 1) = 0.$$

Therefore $(A_G - I)n \in \prod_{G^0} \mathbb{N}$.

We shall now show that for all $v \in G^0$ there exists $w \in G^0$ such that $v \geq w$ and $((A_G - I)n)(w) \geq 1$. If $v \notin L$, then $v \in M$ and $v \geq v_j^k$ for some $v_j^k \in S$. But then there is an edge e_j^k with $s(e_j^k) = v_j^k$ and $r(e_j^k) = v_{j+1}^k \neq v_j^k$. Thus we have that

$$\begin{aligned} ((A_G - I)n)(v_j^k) \geq & a_{v_j^k}(A_G(v_j^k, v_j^k) - 1) + a_{v_{j+1}^k} A_G(v_j^k, v_{j+1}^k) \\ \geq & (j)(0-1) + (j+1)(1) = 1. \end{aligned}$$

On the other hand, if $v \in L$, then v feeds into a loop. Since G satisfies Condition (L) this loop must have an exit. Therefore, there exists $w \in L$ such that $v \geq w$ and w is the source of two distinct edges $e, f \in G^1$, where one of the edges, say e, is the edge of a loop and hence has the property that $r(e) \in L$. Now consider the following three cases.

Case 1: $r(f) \notin L$. Then $r(f) \in M$ and hence $r(f) \geq v_j^k$ for some $v_j^k \in S$. But then $v \geq v_j^k$ and $((A_G - I)n)(v_j^k) \geq 1$ as above.

Case 2: $r(f) \in L$ and $r(e) = r(f)$. Then

$$((A_G - I)n)(w) \geq -a_w + a_{r(f)} A_G(w, r(f)) \geq -1 + (1)(2) = 1.$$

Case 3: $r(f) \in L$ and $r(e) \neq r(f)$. Then

$$\begin{aligned}((A_G - I)n)(w) \geq &- a_w + a_{r(e)} A_G(w, r(e)) + a_{r(f)} A_G(w, r(f)) \\ \geq &- 1 + (1)(1) + (1)(1) = 1.\end{aligned}$$

$\square$

Lemma 4.2.12. *Let G be a row-finite graph with no sinks that satisfies Condition (L). Let $x \in \prod_{G^0} \mathbb{N}$. Then there exists an essential 1-sink extension E of G with the property that $[\omega_E] = [x]$ in $\operatorname{coker}(A_G - I)$.*

Proof. By Lemma 4.2.11 we see that there exists $n \in \prod_{G^0} \mathbb{Z}$ with the property that $(A_G - I)n \in \prod_{G^0} \mathbb{N}$ and for all $v \in G^0$ there exists $w \in G^0$ for which $v \geq w$ and $((A_G - I)n)(w) \geq 1$. Since $x + (A_G - I)n \in \prod_{G^0} \mathbb{N}$ we may let E be a 1-sink extension of G with Wojciech vector $\omega_E = x + (A_G - I)n$. Let v_0 be the sink of E. We shall show that E is essential. Let $v \in G^0$. Then there exists $w \in G^0$ for which $v \geq w$ and $((A_G - I)n) \geq 1$. But then $\omega_E(w) \geq ((A_G - I)n)(w) \geq 1$ and w is a boundary vertex of E. Hence $v \geq w \geq v_0$ and we have shown that $G^0 \geq v_0$. Thus E is essential, and furthermore $[\omega_e] = [x + (A_G - I)n] = [x]$ in $\operatorname{coker}(A_G - I)$. $\square$

Proposition 4.2.13. *Let G be a row-finite graph with no sinks that satisfies Condition (L). The Wojciech map $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ is surjective.*

Proof. If x is any vector in $\prod_{G^0} \mathbb{N}$, then by Lemma 4.2.12 there exists an essential 1-sink extensions E for which $[\omega_E] = [x]$. If τ is the Busby invariant of the extension associated to E, then by Lemma 4.2.10 we have that $\omega(\tau) = [\omega_{E_1}] = [x]$. Thus $[x] \in \operatorname{im} \omega$ for all $x \in \prod_{G^0} \mathbb{N}$.

Now because $C^*(G)$ is separable and nuclear (see Remark A.11.13), it follows from [5, Corollary 15.8.4] that $\operatorname{Ext}(C^*(G))$ is a group. Because $\prod_{G^0} \mathbb{N}$ is the positive cone of $\prod_{G^0} \mathbb{Z}$, and hence generates $\prod_{G^0} \mathbb{Z}$, the fact that $[x] \in \operatorname{im} \omega$ for all $x \in \prod_{G^0} \mathbb{N}$ implies that $\operatorname{im} \omega = \operatorname{coker}(A_G - I)$. $\square$

Corollary 4.2.14. *Let G be a row-finite graph with no sinks that satisfies Condition (L). The map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ is surjective.*

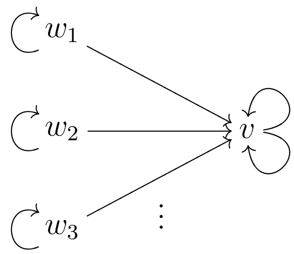
Proof. This follows from the fact that $\omega = \overline{S_G} \circ d$, and $\overline{S_G}$ is an isomorphism. $\square$

Theorem 4.2.15. *Let G be a row-finite graph with no sinks that satisfies Condition (L). The Wojciech map $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ and the Cuntz-Krieger map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ are isomorphisms. Consequently,*

$$\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(A_G - I) \cong \operatorname{coker}(B_G - I).$$

Remark 4.2.16. The above computation of $\operatorname{Ext}(C^*(G))$ has also been extended to the case when G is an arbitrary graph satisfying Condition (L) [23, Theorem 3.1]. A statement of this result appears in Theorem A.12.1 of Appendix A.

Remark 4.2.17. Suppose that G is a row-finite graph with no sinks that satisfies Condition (L), and that τ is an element of $\operatorname{Ext}(C^*(G))$ for which $\omega(\tau) \in \operatorname{coker}(A_G - I)$ can be written as $[x]$ for some $x \in \prod_{G^0} \mathbb{N}$. Then Lemma 4.2.12 shows us that there exists an essential 1-sink extension E with the property that the extension associated to E is equal to τ in $\operatorname{Ext}(C^*(G))$. Thus for every $\tau \in \operatorname{Ext}(C^*(G))$ with the property that $\omega(\tau) = [x]$ for $x \in \prod_{G^0} \mathbb{N}$, we may choose a representative that is the extension associated to an essential 1-sink extension. It is natural to wonder if this is the case for all elements of $\operatorname{Ext}(C^*(G))$. It turns out that in general it is not. To see this let G be the following infinite graph.



Then G is a row-finite graph with no sinks that satisfies Condition (L). However,

$$A_G - I = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix},$$

and if we let $x := \begin{pmatrix} -1 \\ -2 \\ -3 \\ \vdots \end{pmatrix} \in \prod_{G^0} \mathbb{Z}$ then for all $n \in \prod_{G^0} \mathbb{Z}$ we have that

$$x + (A_G - I)n = \begin{pmatrix} -1+n(v) \\ -2+n(v) \\ -3+n(v) \\ \vdots \end{pmatrix}.$$

Thus for any $n \in \prod_{G^0} \mathbb{Z}$ we see that $x + (A_G - I)n$ has negative entries. Hence $x + (A_G - I)n$ cannot be the Wojciech vector of a 1-sink extension for any $n \in \prod_{G^0} \mathbb{Z}$.

It turns out, however, that if we add the condition that G be a finite graph then the result does hold.

Lemma 4.2.18. *Let G be a finite graph with no sinks that satisfies Condition (L). If $v \in G^0$, then there exists $n \in \prod_{G^0} \mathbb{N}$ for which $(A_G - I)n \in \prod_{G^0} \mathbb{N}$ and $((A_G - I)n)(v) \geq 1$.*

Proof. If $A_G(v, v) \geq 2$ then we can let $n = \delta_v$ and the claim holds. Therefore, we shall suppose that $A_G(v, v) \leq 1$. Since G has no sinks and satisfies Condition (L), there must exist an edge $e_1 \in G^1$ with $s(e_1) = v$ and $r(e_1) \neq v$. Then since G has no sinks we may find an edge $e_2 \in G^1$ with $s(e_2) = r(e_1)$, and an edge $e_3 \in G^1$ with $s(e_3) = r(e_2)$. Continuing in this fashion we will produce an infinite path $e_1 e_2 \ldots$ with $s(e_1) = v$. Since G is finite, the vertices $s(e_i)$ of this path must eventually repeat. Let m be the smallest natural number for which $s(e_m) = s(e_k)$ for some $1 \leq k \leq m-1$. Note that because $r(e_1) \neq s(e_1)$ we must have $m \geq 3$.

Now $e_k e_{k+1} \ldots e_{n-1}$ will be a loop, and since G satisfies Condition (L), there exists an exit for this loop. Thus for some $k \leq l \leq n-1$ there exists $f \in G^1$ such that $r(f) = s(e_l)$ and $f \neq e_l$. For each $w \in G^0$ define

$$a_w := \begin{cases} 2 & \text{if } w \in \{s(e_i)\}_{i=2}^{l} \\ 1 & \text{otherwise} \end{cases}$$

Note that $\{s(e_i)\}_{i=2}^{l}$ may be empty. This will occur if and only if $l = 1$. Now let $n := (a_w) \in \prod_{G^0} \mathbb{N}$. To see that $((A_G - I)n)(v) \geq 1$, note that $a_v = 1$, and consider four cases.

Case 1: $l = 1$ and $r(f) = r(e_1)$. Since $r(e_1) \neq v$ we have that

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{r(e_1)} A_G(v, r(e_1)) \geq 1(-1) + 1(2) = 1.$$

Case 2: $l = 1$ and $r(f) = v$. Then

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{r(e_1)} A_G(v, r(e_1)) \geq 1(1-1) + 1(1) = 1.$$

Case 3: $l = 1$, $r(f) \neq r(e_1)$, and $r(f) \neq v$. Then

$$\begin{aligned} ((A_G - I)n)(v) &\geq a_v(A_G(v,v) - 1) + a_{r(e_1)} A_G(v, r(e_1)) + a_{r(f)} A_G(v, r(f)) \\ &\geq 1(-1) + 1(1) + 1(1) \end{aligned}$$

$$= 1.$$

Case 4: $l \geq 2$. Then $a_{r(e_1)} = 2$ and

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{r(e_1)}A_G(v, r(e_1)) \geq 1(-1) + 2(1) = 1.$$

To see that $(A_G - I)n \in \prod_{G^0} \mathbb{N}$ let $w \in G^0$ and consider the following three cases.

Case 1: $w = s(e_l)$ and $r(e_l) = r(f)$. Then $a_w = 2$ and we have

$$((A_G - I)n)(w) \geq a_w(A_G(w,w) - 1) + a_{r(e_l)}A_G(w, r(e_l)) \geq 2(-1) + 1(2) = 0.$$

Case 2: $w = s(e_l)$ and $r(e_l) \neq r(f)$. Then

$$\begin{aligned}((A_G - I)n)(w) &\geq a_w(A_G(w,w) - 1) + a_{r(e_l)}A_G(w, r(e_l)) + a_{r(f)}A_G(w, r(f)) \\ &\geq 2(-1) + 1(1) + 1(1) \\ &= 0.\end{aligned}$$

Case 3: $w \neq s(e_l)$. Then either $w \in \{s(e_i)\}_{i=2}^{l-1}$ or $a_w = 1$. In either case there exists an edge e with $s(e) = w$ and $a_{r(e)} \geq a_w$. Thus

$$((A_G - I)n)(w) \geq a_w(A_G(w,w) - 1) + a_{r(e)}A_G(w, r(e)) \geq -a_w + a_{r(e)} \geq 0$$

and $(A_G - I)n \in \prod_{G^0} \mathbb{N}$. $\square$

Theorem 4.2.19. *Let G be a finite graph with no sinks that satisfies Condition (L). For any $[x] \in \operatorname{coker}(A_G - I)$ there exists an essential 1-sink extension E of G such that $[\omega_E] = [x]$ in $\operatorname{coker}(A_G - I)$.*

Proof. For each $v \in G^0$ we may use Lemma 4.2.18 to obtain a vector $n_v \in \prod_{G^0} \mathbb{N}$ such that $(A_G - I)n_v \in \prod_{G^0} \mathbb{N}$ and $((A_G - I)n_v)(v) \geq 1$. Now write x in the form $x = \sum_{v \in G^0} a_v \delta_v$. Let $n := \sum_{v \in G^0}(|a_v|+1)n_v$. Then by linearity, $x+(A_G-I)n \in \prod_{G^0} \mathbb{N}$ and $x + (A_G - I)n \neq 0$. Let E be a 1-sink extension of G with sink v_0 and Wojciech vector equal to $x + (A_G - I)n$. Then $[\omega_E] = [x + (A_G - I)n] = [x]$ in $\operatorname{coker}(A_G - I)$. Furthermore, since $\omega_E(v) \geq 1$ for all $v \in G^0$ it follows that $G^0 \geq v_0$ and E is an essential 1-sink extension. □

This result shows that if G is a finite graph with no sinks that satisfies Condition (L), then for any element in $\operatorname{Ext}(C^*(G))$ we may choose a representative that is the extension associated to an essential 1-sink extension E of G. Furthermore, since the Wojciech map is an isomorphism we see that if E_1 and E_2 are essential 1-sink extensions that are representatives for $\tau_1, \tau_2 \in \operatorname{Ext}(C^*(G))$, then the essential 1-sink extension with Wojciech vector equal to $\omega_{E_1} + \omega_{E_2}$ will be a representative of $\tau_1 + \tau_2$. Hence we have a way of choosing representatives of the classes in Ext that have a nice visual interpretation and for which we can easily compute their sum.

4.3 Semiprojectivity of graph algebras

In 1983 Effros and Kaminker [24] began the development of a shape theory for C^*-algebras that generalized the topological theory. In their work they looked at C^*-algebras with a property that they called semiprojectivity. These semiprojective C^*-algebras are the noncommutative analogues of absolute neighborhood retracts. In 1985 Blackadar generalized many of these results [6], but because he wished to apply shape theory to C^*-algebras not included in [24] and because the theory in [24] was not a direct noncommutative generalization, Blackadar gave a new definition of semiprojectivity. Blackadar's definition is more restrictive than that in [24].

Definition 4.3.1 *(Blackadar).* A separable C^*-algebra A is *semiprojective* if for any C^*-algebra B, any increasing sequence $\{J_n\}_{n=1}^{\infty}$ of (closed two-sided) ideals, and any $*$-homomorphism $\phi : A \to B/J$, where $J := \overline{\bigcup_{n=1}^{\infty} J_n}$, there is an n and a $*$-homomorphism $\psi : A \to B/J_n$ such that

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr, "\phi"'] & B/J_n \arrow[d, "\pi"] \\ & B/J \end{tikzcd}$$

where $\pi : B/J_n \to B/J$ is the natural quotient map.

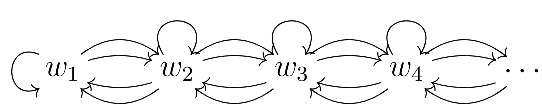
In [6] it was shown that the Cuntz-Krieger algebras are semiprojective, and more recently Blackadar has announced a proof that $\mathcal{O}_\infty$ is semiprojective. Based on the proof for $\mathcal{O}_\infty$ Szymański has proven in [92] that if E is a transitive graph with finitely many vertices (but a possibly infinite number of edges), then $C^*(E)$ is semiprojective.

We now give an example of a row-finite transitive graph G with an infinite number of vertices and with the property that $C^*(G)$ is not semiprojective. We use the fact that the Wojciech map of §4.2 is an isomorphism in order to prove that $C^*(G)$ is not semiprojective.

If G is a graph, then by *adding a sink at* $v \in G^0$ we shall mean adding a single vertex v_0 to G^0 and a single edge e to G^1 going from v to v_0. More formally, if G is a graph, then we form the graph F defined by $F^0 := G^0 \cup \{v_0\}$, $F^1 := G^1 \cup \{e\}$, and we extend r and s to F^1 by defining and $r(e) = v_0$ and $s(e) = v$.

Example 4.3.2.

G

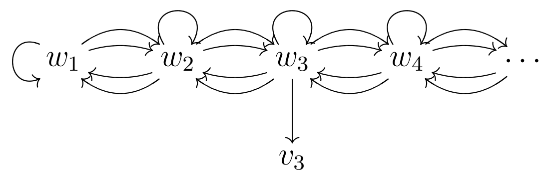


If G is the above graph, then note that G is transitive, row-finite, and has no sinks.

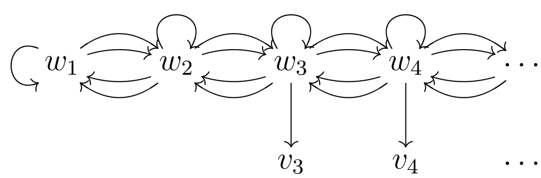
Theorem 4.3.3. *If G is the graph in Example 4.3.2, then $C^*(G)$ is not semiprojective.*

Proof. For each $i \in \mathbb{N}$ let E_i be the graph formed by adding a sink to G at w_i, and let F_i be the graph formed by adding a sink to each vertex in $\{w_i, w_{i+1}, \ldots\}$. In each case we shall let v_i denote the sink that is added at w_i. As examples we draw E_3 and F_3:

E_3



F_3



We shall now assume that $C^*(G)$ is semiprojective and arrive at a contradiction. Let $B := C^*(F_1)$ and for each $n \in \mathbb{N}$ let $H_n := \{v_1, v_2, \ldots, v_n\}$. Also let $H_\infty := \{v_1, v_2, \ldots\}$. Set $J_n := I_{H_n}$. Then $\{J_n\}_{n=1}^\infty$ is an increasing sequence of ideals and $J := \bigcup_{n=1}^\infty J_n = I_{H_\infty}$. Now $B/J = C^*(F_1)/I_{H_\infty} \cong C^*(G)$ and for each $n \in \mathbb{N}$, $B/J_n \cong C^*(F_{n+1})$ by [4, Theorem 4.1]. Thus if we identify $C^*(G)$ and B/J, then by semiprojectivity there exists a homomorphism $\psi : C^*(G) \to B/J_n$ for some n

$$\begin{tikzcd} C^*(G) \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}"'] & B/J_n \cong C^*(F_{n+1}) \arrow[d, "\pi"] \\ & B/J \cong C^*(G) \end{tikzcd}$$

such that $\pi \circ \psi = \mathrm{id}$. Note that the projection $\pi : B/J_n \to B/J$ is just the projection $\pi : C^*(F_{n+1}) \to C^*(F_{n+1})/I_{\{v_{n+1}, v_{n+2}, \ldots\}} \cong C^*(G)$.

Now if we let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger F_{n+1}-family in $C^*(F_{n+1})$

and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E_{n+1}-family in $C^*(E_{n+1})$, then by the universal property of $C^*(F_{n+1})$ there exists a homomorphism $\rho : C^*(F_{n+1}) \to C^*(E_{n+1})$ such that

$$\rho(s_e) = \begin{cases} t_e & \text{if } e \in E^1_{n+1} \\ 0 & \text{if } e \in F^1_{n+1} \backslash E^1_{n+1} \end{cases} \qquad \text{and} \qquad \rho(p_v) = \begin{cases} q_v & \text{if } v \in E^0_{n+1} \\ 0 & \text{if } v \in F^0_{n+1} \backslash E^0_{n+1}. \end{cases}$$

Since E_{n+1} is a 1-sink extension of G, we have the usual projection $\pi_{E_{n+1}} : C^*(E_{n+1}) \to C^*(G)$. One can then check that the diagram

$$\begin{tikzcd} C^*(F_{n+1}) \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & C^*(E_{n+1}) \arrow[dl, "\pi_{E_{n+1}}"] \\ & C^*(G) & \end{tikzcd}$$

commutes simply by checking that $\pi_{E_{n+1}} \circ \rho$ and π agree on generators. This, combined with the fact that $\pi \circ \psi = \mathrm{id}$ on $C^*(G)$, implies that $\pi_{E_{n+1}} \circ \rho \circ \psi = \mathrm{id}$. Hence the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I_{v_{n+1}} \arrow[r] & C^*(E_{n+1}) \arrow[r, "\pi_{E_{n+1}}"] & C^*(G) \arrow[r] \arrow[l, bend right, "\rho\circ\psi"'] & 0 \end{tikzcd}$$

is split exact. Therefore this extension is degenerate. Since $I_{v_{n+1}} \cong \mathcal{K}$ by [54, Corollary 2.2] we have that this extension is in the zero class in $\mathrm{Ext}(C^*(G))$.

However, the Wojciech vector of E_{n+1} is $\omega_{E_{n+1}} = \delta_{w_{n+1}}$. Since

$$A_G - I = \begin{pmatrix} 0 & 2 & 0 & 0 & \cdots \\ 2 & 0 & 2 & 0 & \\ 0 & 2 & 0 & 2 & \\ 0 & 0 & 2 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

we see that every vector in the image of $A_G - I$ has entries that are multiples of 2. Thus

$\delta_{w_{n+1}} \notin \operatorname{im}(A_G - I)$, and $[\omega_{E_{n+1}}]$ is not zero in $\operatorname{coker}(A_G - I)$. But then Proposition 4.2.10 and Theorem 4.2.15 imply that the extension associated to $C^*(E_{n+1})$ is not equal to zero in $\operatorname{Ext}(C^*(G))$. This provides the contradiction, and hence $C^*(G)$ cannot be semiprojective. $\square$

Remark 4.3.4. After the completion of this work, Spielberg proved in [90] that all classifiable, simple, separable, purely infinite C^*-algebras having finitely generated K-theory and torsion-free K_1-group are semiprojective [90, Theorem 3.12]. This was accomplished by realizing certain C^*-algebras as graph algebras of transitive graphs. It also implies that if G is a transitive graph that is not a single loop, and if $C^*(G)$ has finitely generated K-theory and torsion-free K_1-group, then $C^*(G)$ is semiprojective. We mention that if one computes the K-theory of the C^*-algebra associated to the graph in Example 4.3.2, using [78, Theorem 3.2] for instance, one sees that it is not finitely generated.

Chapter 5

$C^*(G)$-embeddability

In Chapter 2 we looked at a fixed row-finite graph G and considered *1-sink extensions* of G formed by adding a single sink v_0 to G. We also saw that for any 1-sink extension E of G there is an exact sequence

$$0 \longrightarrow I_{v_0} \xrightarrow{\ i\ } C^*(E) \xrightarrow{\ \pi_E\ } C^*(G) \longrightarrow 0. \tag{5.1}$$

Here I_{v_0} denotes the ideal generated by the projection p_{v_0} corresponding to the sink v_0. Recall that if E_1 and E_2 are 1-sink extensions, then we say that $C^*(E_2)$ may be $C^*(G)$-*embedded* into $C^*(E_1)$ if $C^*(E_2)$ is isomorphic to a full corner of $C^*(E_1)$ via an isomorphism which commutes with the π_{E_i}'s.

It was shown in Chapter 2 that $C^*(G)$-embeddability of 1-sink extensions is determined by the class of the Wojciech vector in $\operatorname{coker}(A_G - I)$, where A_G is the vertex matrix of G. Specifically, it was shown in Theorem 2.2.3 that if G is a graph with no sinks or sources, (E_1, v_1) and (E_2, v_2) are two essential 1-sink extensions of G whose Wojciech vectors have only a finite number of nonzero entries, and ω_{E_1} and ω_{E_2} are in the same class in $\operatorname{coker}(A_G - I)$, then there exists a 1-sink extension F of G such that

$C^*(F)$ may be $C^*(G)$-embedded in both $C^*(E_1)$ and $C^*(E_2)$. In addition, a version of this result was proven for non-essential 1-sink extensions in Proposition 2.3.3 and a partial converse for both results was obtained in Corollary 2.5.4. In this chapter we show that when every loop in G has an exit, much stronger results hold.

We shall see in §5.1 that if (E, v_0) is a 1-sink extension of G, then (except in degenerate cases) we will have $I_{v_0} \cong \mathcal{K}$. Thus we see from (5.1) that $C^*(E)$ is an extension of $C^*(G)$ by the compact operators. Hence, E determines an element in $\operatorname{Ext}(C^*(G))$. In §5.1 we prove the following.

Theorem. *Let G be a row-finite graph and (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. Then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if E_1 and E_2 determine the same element in* $\operatorname{Ext}(C^*(G))$.

It was shown in Chapter 4 that if G is a graph in which every loop has an exit, then $\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(A_G - I)$. Using the isomorphism constructed there we are able to translate the above result into a statement about the Wojciech vectors. Specifically we prove the following.

Theorem. *Let G be a row-finite graph in which every loop has an exit. If (E_1, v_1) and (E_2, v_2) are essential 1-sink extensions of G, then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if $[\omega_{E_1}] = [\omega_{E_2}]$ in* $\operatorname{coker}(A_G - I)$.

Provided that one is willing to allow all the loops in G to have exits, this result is an improvement over Theorem 2.2.3 in the following respects. First of all, G is allowed to have sources and there are no conditions on the Wojciech vectors of E_1 and E_2. Second, we see that the graph F in the statement of Theorem 2.2.3 can actually be chosen to be either E_1 or E_2. And finally, we see that the equality of the Wojciech vectors in $\operatorname{coker}(A_G - I)$ is not only sufficient but necessary. In §5.3 we obtain a version of this theorem for non-essential extensions.

This chapter is organized as follows. In §5.1 we show how to associate an element of $\operatorname{Ext}(C^*(G))$ to a (not necessarily essential) 1-sink extension. We then prove that if E_1 and E_2 are 1-sink extensions of G, then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if E_1 and E_2 determine the same element in $\operatorname{Ext}(C^*(G))$. In §5.2 we use the isomorphism $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ from Chapter 4 to prove that for essential 1-sink extensions, $C^*(G)$-embeddability may be characterized in terms of the Wojciech vector. In §5.3 we discuss non-essential extensions and again use the isomorphism ω to obtain a characterization of $C^*(G)$-embeddability for arbitrary 1-sink extensions. We conclude with an example and some observations

5.1 $C^*(G)$-embeddability and CK-equivalence

In this chapter we will follow the approach in §3.2 and view Ext as the CK-equivalence classes of essential extensions. Recall that if E is a 1-sink extension of G with sink v_0, then it follows from [54, Corollary 2.2] that $I_{v_0} \cong \mathcal{K}(\ell^2(E^*(v_0)))$ where $E^*(v_0) = \{\alpha \in E^* : r(\alpha) = v_0\}$. Thus $I_{v_0} \cong \mathcal{K}$ when $E^*(v_0)$ contains infinitely many elements, and $I_{v_0} \cong M_n(\mathbb{C})$ when $E^*(v_0)$ contains a finite number of elements. If G has no sources, then it is easy to see that $E^*(v_0)$ must have infinitely many elements, and it was shown in Lemma 4.2.5 that if E is an essential 1-sink extension of G, then $E^*(v_0)$ will also have infinitely many elements. Consequently, in each of these cases we will have $I_{v_0} \cong \mathcal{K}$. Furthermore, one can see from the proof of [54, Corollary 2.2] that p_{v_0} is a minimal projection in I_{v_0}.

Definition 5.1.1. Let G be a row-finite graph and let (E, v_0) be a 1-sink extension of G. If $I_{v_0} \cong \mathcal{K}$, (i.e. $E^*(v_0)$ has infinitely many elements), then choose any isomorphism $i_E : \mathcal{K} \to I_{v_0}$, and define the *extension associated to* E to be (the strong equivalence class of) the Busby invariant $\tau : C^*(G) \to \mathcal{Q}$ associated to the short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_E} C^*(E) \xrightarrow{\pi_E} C^*(G) \longrightarrow 0.$$

If $I_{v_0} \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ (i.e. $E^*(v_0)$ has finitely many elements), then the *extension associated to* E is defined to be (the strong equivalence class of) the zero map $\tau : C^*(G) \to \mathcal{Q}$. That is, $\tau : C^*(G) \to \mathcal{Q}$ and $\tau(x) = 0$ for all $x \in C^*(G)$.

Note that the extension associated to E is always a map from $C^*(G)$ into $\mathcal{Q}$. Also note that the above definition is well-defined in the case when $I_{v_0} \cong \mathcal{K}$. That is, two different choices of i_E will produce extensions with strongly equivalent Busby invariants (see [102, Problem 3E(c)] for more details). Also, since p_{v_0} is a minimal projection, $i_E^{-1}(p_{v_0})$ will always be a rank 1 projection.

Our goal in the remainder of this section is to prove the following theorem and its corollary.

Theorem 5.1.2. *Let G be a row-finite graph, and let E_1 and E_2 be 1-sink extensions of G. Then the extensions associated to E_1 and E_2 are CK-equivalent if and only if one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.*

Corollary 5.1.3. *Let G be a row-finite graph, and let E_1 and E_2 be essential 1-sink extensions of G. Then the extensions associated to E_1 and E_2 are equal in* $\operatorname{Ext}(C^*(G))$ *if and only if one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.*

Remark 5.1.4. Note that we are not assuming that each of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other, only that one of them can.

Proof of Corollary 5.1.3 . Because E_1 and E_2 are essential it follows from Lemma 4.2.5 that $I_{v_1} \cong I_{v_2} \cong \mathcal{K}$. Furthermore, Lemma 3.2.3 shows that two essential extensions are equal in Ext if and only if they are CK-equivalent. $\square$

Lemma 5.1.5. *Let P and Q be rank 1 projections in $\mathcal{B}$. Then there exists a unitary $U \in \mathcal{B}$ such that $P = U^*QU$ and $I - U$ has finite rank.*

Proof. Straightforward. $\square$

Lemma 5.1.6. *Let G be a row-finite graph, and let (E, v_0) be a 1-sink extension of G. If the extension associated to E is the zero map, then there is an isomorphism $\Psi : C^*(E) \to C^*(G) \oplus I_{v_0}$ which makes the diagram*

$$\begin{tikzcd} C^*(E) \arrow[r, "\Psi"] \arrow[dr, "\pi_E"'] & C^*(G) \oplus I_{v_0} \arrow[d, "p_1"] \\ & C^*(G) \end{tikzcd}$$

commute. Here p_1 is the projection $(a, b) \mapsto a$.

Proof. Since the extension associated to E is zero, one of two things must occur. If $I_{v_0} \cong \mathcal{K}$, then τ is the Busby invariant of $0 \to I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \to 0$. If $I_{v_0} \cong M_n(\mathbb{C})$, then since $M_n(\mathbb{C})$ is unital it follows that $\mathcal{Q}(I_{v_0}) \cong \mathcal{M}(M_n(\mathbb{C}))/M_n(\mathbb{C}) = 0$ and the Busby invariant of $0 \to I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \to 0$ must be the zero map. In either case, the Busby invariant of the extension $0 \to I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \to 0$ is zero. From [102, Proposition 3.2.15] it follows that $C^*(E) \cong C^*(G) \oplus I_{v_0}$ via the map $\Psi(x) := (\pi_E(x), \sigma(x))$, where $\sigma : C^*(E) \to I_{v_0}$ denotes the (unique) map for which $\sigma \circ i$ is the identity. The fact that $p_1 \circ \Psi = \pi_E$ then follows by checking each on generators of $C^*(E)$. $\square$

Proof of Sufficiency in Theorem 5.1.2 . Let E_1 and E_2 are 1-sink extensions of G whose associated extensions are CK-equivalent. Also let v_1 and v_2 denote the sinks of E_1 and E_2 and τ_1 and τ_2 be the extensions associated to E_1 and E_2. Consider the following cases.

CASE 1: Either $E^*(v_1)$ is finite or $E^*(v_2)$ is finite.

Without loss of generality let us assume that $E^*(v_1)$ is finite and the number of elements in $E^*(v_1)$ is less than or equal to the number of elements in $E^*(v_2)$. Then $I_{v_1} \cong M_n(\mathbb{C})$ for some finite n, and because $I_{v_2} \cong \mathcal{K}(\ell^2(E^*(v_2)))$ we see that either $I_{v_2} \cong \mathcal{K}$ or $I_{v_2} \cong M_m(\mathbb{C})$ for some finite $m \geq n$. In either case we may choose an imbedding $\phi : I_{v_1} \to I_{v_2}$ which maps onto a full corner of I_{v_2}. (Note that since I_{v_2} is simple we need only choose ϕ to map onto a corner, and then that corner is automatically full.) Furthermore, since p_{v_1} and q_{v_2} are rank 1 projections, we may choose ϕ in such a way that $\phi(p_{v_1}) = q_{v_2}$. We now define $\Phi : C^*(G) \oplus I_{v_1} \to C^*(G) \oplus I_{v_2}$ by $\Phi((a,b)) = (a, \phi(b))$. We see that Φ maps $C^*(G) \oplus I_{v_1}$ onto a full corner of $C^*(G) \oplus I_{v_2}$ and that Φ makes the diagram

$$\begin{tikzcd}
C^*(G) \oplus I_{v_1} \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & C^*(G) \oplus I_{v_2} \arrow[dl, "p_1"] \\
& C^*(G) &
\end{tikzcd}$$

commute, where p_1 is the projection $(a,b) \mapsto a$. Now since $\tau_1 = 0$ and τ_2 is CK-equivalent to τ_2, it follows that $\tau_2 = 0$. Thus Lemma 5.1.6, the existence of Φ, and the above commutative diagram imply that $C^*(E_1)$ is $C^*(G)$-embeddable into $C^*(E_2)$.

CASE 2: Both $E^*(v_1)$ and $E^*(v_2)$ are infinite.

Then $I_{v_1} \cong I_{v_2} \cong \mathcal{K}$. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E_1-family in $C^*(E_1)$, and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E_2-family in $C^*(E_2)$. For $k \in \{1,2\}$, choose isomorphisms $i_{E_k} : \mathcal{K} \to I_{v_k}$ so that the Busby invariant τ_k of

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_{E_k}} C^*(E_k) \xrightarrow{\pi_{E_k}} C^*(G) \longrightarrow 0$$

is an extension associated to E_k. By hypothesis τ_1 and τ_2 are CK-equivalent. Therefore, after interchanging the roles of E_1 and E_2 if necessary, we may assume that there exists an isometry $W \in \mathcal{B}$ for which $\tau_1 = \mathrm{Ad}(\pi(W)) \circ \tau_2$ and $\tau_2 = \mathrm{Ad}(\pi(W^*)) \circ \tau_1$.

For $k \in \{1,2\}$, let $PB_k := \{(T,a) \in \mathcal{B} \oplus C^*(G) : \pi(T) = \tau_k(a)\}$ be the pullback C^*-algebra along π and τ_k. It follows from [102, Proposition 3.2.11] that $PB_k \cong C^*(E_k)$. Now for $k \in \{1,2\}$, let σ_k be the unique map which makes the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "i_{E_k}"] \arrow[dr, "i"'] & C^*(E_k) \arrow[d, "\sigma_k"] \\ & \mathcal{B} \end{tikzcd}$$

commute. Then $\sigma_1(p_{v_1})$ and $\sigma_2(q_{v_2})$ are rank 1 projections in $\mathcal{B}$. Choose a unit vector $x \in (\ker W^*)^{\perp}$. By Lemma 5.1.5 there exists a unitary $U_1 \in \mathcal{B}$ such that $U_1 \sigma_2(q_{v_2}) U_1^*$ is the projection onto $\operatorname{span}\{x\}$, and for which $I - U_1$ is compact. Therefore, by the way in which x was chosen $WU_1\sigma_2(q_{v_2})U_1^*W^*$ is a rank 1 projection. We may then use Lemma 5.1.5 again to produce a unitary $U_2 \in \mathcal{B}$ for which $U_2(WU_1\sigma_2(q_{v_2})U_1^*W^*)U_2^* = \sigma_1(p_{v_1})$, and $I - U_2$ is compact.

Let $V := U_2WU_1$. Then V is an isometry, and we may define a map $\Phi : PB_2 \to PB_1$ by $\Phi((T,a)) = (VTV^*, a)$. Since $V^*V = I$ it follows that Φ is a homomorphism, and since U_1 and U_2 differ from I by a compact operator, we see that $\pi(V) = \pi(W)$. Therefore

$$\pi(VTV^*) = \pi(W)\pi(T)\pi(W^*) = \pi(W)\tau_2(a)\pi(W^*) = \tau_1(a)$$

so $(VTV^*, a) \in PB_1$, and Φ does in fact take values in PB_1.

For $k \in \{1,2\}$, let $p_k : PB_k \to C^*(G)$ be the projection $p_k((T,a)) = a$. Then the diagram

$$\begin{tikzcd}
PB_2 \arrow[rr, "\Phi"] \arrow[dr, "p_2"'] & & PB_1 \arrow[dl, "p_1"] \\
& C^*(G) &
\end{tikzcd}$$

commutes and $\Phi((\sigma_2(q_{v_2}), 0)) = (\sigma_1(p_{v_1}), 0)$. Also, for $k \in \{1, 2\}$, let Ψ_k be the standard isomorphism from $C^*(E_k)$ to PB_k given by $\Psi_k(x) = (\sigma_1(x), \pi_{E_k}(x))$ [102, Proposition 3.2.11]. Then for each $k \in \{1, 2\}$, the diagram

$$\begin{tikzcd}
C^*(E_k) \arrow[rr, "\Psi_k"] \arrow[dr, "\pi_{E_k}"'] & & PB_k \arrow[dl, "p_k"] \\
& C^*(G) &
\end{tikzcd}$$

commutes and we have that $\Psi_1(p_{v_1}) = (\sigma_1(p_{v_1}), 0)$ and $\Psi_2(q_{v_2}) = (\sigma_2(q_{v_2}), 0)$. If we define $\phi : C^*(E_2) \to C^*(E_1)$ by $\phi := \Psi_1^{-1} \circ \Phi \circ \Psi_2$, then the diagram

$$\begin{tikzcd}
C^*(E_2) \arrow[rr, "\phi"] \arrow[dr, "\pi_{E_2}"'] & & C^*(E_1) \arrow[dl, "\pi_{E_1}"] \\
& C^*(G) &
\end{tikzcd} \tag{5.2}$$

commutes and $\phi(q_{v_2}) = p_{v_1}$.

We shall now show that ϕ embeds $C^*(E_2)$ onto a full corner of $C^*(E_1)$. We begin by showing that Φ embeds PB_2 onto a corner of PB_1. To see that Φ is injective, note that since V is an isometry

$$\begin{aligned}
\|VTV^*\|^2 = \|(VTV^*)(VTV^*)^*\| = \|VTV^*VT^*V^*\| = \|VTT^*V^*\| \\
= \|(VT)(VT)^*\| = \|VT\|^2 = \|T\|^2.
\end{aligned}$$

Therefore $\|VTV^*\| = \|T\|$, and

$$\|\Phi((T,a))\| = \|(VTV^*,a)\| = \max\{\|VTV^*\|,\|a\|\} = \max\{\|T\|,\|a\|\} = \|(T,a)\|.$$

Next we shall show that the image of Φ is a corner in PB_1. Let $P := VV^*$ be the range projection of V. We shall define a map $L_P : PB_1 \to PB_1$ by $L_P((T,a)) = (PT,a)$. To see that L_P actually takes values in PB_1 recall that U_1 and U_2 differ from I by a compact operator and therefore $\pi(V) = \pi(W)$. We then have that

$$\begin{aligned}\pi(PT) &= \pi(VV^*)\pi(T) = \pi(WW^*)\tau_1(a) = \pi(WW^*)\pi(W)\tau_2(a)\pi(W^*) \\ &= \pi(W)\tau_2(a)\pi(W^*) = \tau_1(a).\end{aligned}$$

Hence $(PT,a) \in PB_1$. In a similar way we may define $R_P : PB_1 \to PB_1$ by $R_P((T,a)) = (TP,a)$. Since P is a projection, we see that L_P and R_P are bounded linear maps. One can also check that (L_P, R_P) is a double centralizer and therefore defines an element $\mathcal{P} := (L_P, R_P) \in \mathcal{M}(PB_1)$. Because P is a projection, $\mathcal{P}$ must also be a projection. Also for any $(T,a) \in PB_1$ we have that $\mathcal{P}(T,a) = (PT,a)$ and $(T,a)\mathcal{P} = (TP,a)$.

Now for all $(T,a) \in PB_2$ we have

$$\begin{aligned}\Phi((T,a)) &= (VTV^*,a) = (VV^*VTV^*VV^*,a) \\ &= (PVTV^*P,a) = \mathcal{P}(VTV^*,a)\mathcal{P} = \mathcal{P}\Phi((T,a))\mathcal{P}\end{aligned}$$

and therefore Φ maps PB_2 into the corner $\mathcal{P}(PB_1)\mathcal{P}$. We shall now show that Φ actually maps onto this corner. If $(T,a) \in \mathcal{P}(PB_1)\mathcal{P}$, then

$$\pi(V^*TV) = \pi(W)^*\pi(T)\pi(W) = \pi(W)^*\tau_1(a)\pi(W) = \tau_2(a)$$

and so $(VTV^*, a) \in PB_2$. But then $\Phi((V^*TV, a)) = (VV^*TVV^*, a) = (PTP, a) = \mathcal{P}(T, a)\mathcal{P} = (T, a)$. Thus Φ embeds PB_2 onto the corner $\mathcal{P}(PB_1)\mathcal{P}$.

Because Ψ_1 and Ψ_2 are isomorphisms, it follows that ϕ embeds $C^*(E_2)$ onto a corner of $C^*(E_1)$. We shall now show that this corner must be full. This will follow from the commutativity of diagram 5.2. Let I be any ideal in $C^*(E_1)$ with the property that $\operatorname{im}\phi \subseteq I$. Since $\phi(q_{v_2}) = p_{v_1}$ it follows that $p_{v_1} \in \operatorname{im}\phi \subseteq I$. Therefore, $I_{v_1} \subseteq I$. Furthermore, for any $w \in G^0$ we have by commutativity that $\pi_{E_1}(p_w - \phi(q_w)) = 0$. Therefore $p_w - \phi(q_w) \in \ker \pi_{E_1} = I_{v_1}$, and it follows that $p_w - \phi(q_w) \in I_{v_1} \subseteq I$. Since $\phi(q_w) \in \operatorname{im}\phi \subseteq I$, this implies that $p_w \in I$ for all $w \in G^0$. Thus $p_w \in I$ for all $w \in G^0 \cup \{v_1\}$. If we let $H := \{v \in E_1^0 : p_v \in I\}$, then it follows from [4, Lemma 4.2] that H is a saturated hereditary subset of $C^*(E_1)$. Since we see from above that H contains $G^0 \cup \{v_1\}$, and since E_1 is a 1-sink extension of G, it follows that $H = E_1^0$. Therefore $I_H = C^*(E_1)$ and since $I_H \subseteq I$ it follows that $I = C^*(E_1)$. Hence $\operatorname{im}\phi$ is a full corner in $C^*(E_1)$. □

Proof of Necessity in Theorem 5.1.2 . Let E_1 and E_2 be 1-sink extensions of G and suppose that $C^*(E_2)$ is $C^*(G)$-embeddable into $C^*(E_1)$. Let v_1 and v_2 denote the sinks of E_1 and E_2, respectively. For $k \in \{1, 2\}$ let $E_k^*(v_k) := \{\alpha \in E_k^* : r(\alpha) = v_k\}$, and let $\phi : C^*(E_2) \to C^*(E_1)$ be a $C^*(G)$-embedding. Consider the following cases.

Case 1: $E_1^*(v_1)$ is finite.

Then $I_{v_1} \cong M_n(\mathbb{C})$ for some finite n. Since $\phi(I_{v_2}) \subseteq I_{v_1}$, and $I_{v_2} \cong \mathcal{K}(\ell^2(E_2^*(v_2)))$, a dimension argument implies that $E_2^*(v_2)$ must be finite. Thus if τ_1 and τ_2 are the extensions associated to E_1 and E_2, we have that $\tau_1 = \tau_2 = 0$ so that τ_1 and τ_2 are CK-equivalent.

Case 2: $E_1^*(v_1)$ is infinite.

Then $I_{v_1} \cong \mathcal{K}$. Choose any isomorphism $i_{E_1} : \mathcal{K} \to I_{v_1}$, and let $\sigma : C^*(E_1) \to \mathcal{B}$

be the (unique) map which makes the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "i_{E_1}"] \arrow[dr, "i"'] & C^*(E_1) \arrow[d, "\sigma"] \\ & \mathcal{B} \end{tikzcd}$$

commute. If we let τ_1 be the corresponding Busby invariant, then τ_1 is the extension associated to E_1.

Furthermore, we know that $I_{v_2} \cong \mathcal{K}(H)$, where H is a Hilbert space which is finite-dimensional if $E_2^*(v_2)$ is finite and infinite-dimensional if $E_2^*(v_2)$ is infinite. Choose an isomorphism $i_{E_2} : \mathcal{K}(H) \to I_{v_2}$. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}(H) \arrow[r, "i_{E_2}"] & C^*(E_2) \arrow[r, "\pi_{E_2}"] \arrow[d, "\phi"] & C^*(G) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "i_{E_1}"] \arrow[d, equal] & C^*(E_1) \arrow[r, "\pi_{E_1}"] \arrow[d, "\sigma"] & C^*(G) \arrow[r] \arrow[d, "\tau_1"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & \mathcal{B} \arrow[r, "\pi"] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd} \tag{5.3}$$

commutes and has exact rows.

Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E_2-family in $C^*(E_2)$ and $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E_1-family in $C^*(E_1)$.

We shall now define a bounded linear transformation $U : H \to \mathcal{H}$. Since $i_{E_2}^{-1}(p_{v_2})$ is a rank 1 projection, we may write $i_{E_2}^{-1}(p_{v_2}) = e \otimes e$, where e is a unit vector in $\operatorname{im} i_{E_2}^{-1}(p_{v_2})$. Likewise, we may write $i_{E_1}^{-1}(q_{v_1}) = f \otimes f$ for some unit vector $f \in \operatorname{im} i_{E_1}^{-1}(q_{v_1})$. For convenience of notation write $\beta := \sigma \circ \phi \circ i_{E_2}$. Note that $\phi(p_{v_2}) = q_{v_1}$ implies that $\beta(e \otimes e) = f \otimes f$. Now for any $h \in H$ define

$$U(h) := \beta(h \otimes e)(f).$$

Then U is a linear transformation and

$$\begin{aligned}\langle U(h), U(k)\rangle &= \langle \beta(h\otimes e)(f), \beta(k\otimes e)(f)\rangle = \langle \beta(k\otimes e)^*\beta(h\otimes e)(f), f\rangle \\ &= \langle \beta(\langle h,k\rangle(e\otimes e))(f), f\rangle = \langle h,k\rangle\langle \beta(e\otimes e)(f), f\rangle \\ &= \langle h,k\rangle\langle (f\otimes f)(f), f\rangle = \langle h,k\rangle\langle f,f\rangle = \langle h,k\rangle.\end{aligned}$$

Therefore U is an isometry.

Now since ϕ embeds $C^*(E_2)$ onto a full corner of $C^*(E_1)$, it follows that there exists a projection $p \in \mathcal{M}(C^*(E_1))$ such that $\operatorname{im}\phi = pC^*(E_1)p$. Because σ is a nondegenerate representation (since $\sigma(I_{v_1}) = \mathcal{K}$), it extends to a representation $\overline{\sigma} : \mathcal{M}(C^*(E_1)) \to \mathcal{B}$ by [80, Corollary 2.51]. Let $P := \overline{\sigma}(p)$. We shall show that $\operatorname{im}P \subseteq \operatorname{im}U$. Let $g \in \operatorname{im}P$. Also let f be as before. Then $g \otimes f \in \mathcal{K}$ and

$$\sigma(pi_{E_1}(g\otimes f)p) = \overline{\sigma}(p)\sigma(i_{E_1}(g\otimes f))\overline{\sigma}(p) = P(g\otimes f)P = \sigma(i_{E_1}(g\otimes f)).$$

Now since $pi_{E_1}(g\otimes f)p \in pC^*(E_1)p = \operatorname{im}\phi$, there exists $a \in C^*(E_2)$ such that $\phi(a) = pi_{E_1}(g\otimes f)p$. In addition, since $\pi_{E_1} : C^*(E_1) \to C^*(G)$ is surjective, it extends to a homomorphism $\overline{\pi}_{E_1} : \mathcal{M}(C^*(E_1)) \to \mathcal{M}(C^*(G))$ by [80, Corollary 2.51]. By commutativity and exactness we then have that

$$\pi_{E_2}(a) = \pi_{E_1}(\phi(a)) = \pi_{E_1}(pi_{E_1}(g\otimes f)p) = \overline{\pi}_{E_1}(p)\pi_{E_1}(i_{E_1}(g\otimes f))\overline{\pi}_{E_1}(p) = 0.$$

Thus $a \in \operatorname{im} i_{E_2}$ by exactness, and we have that $a = i_{E_2}(T)$ for some $T \in \mathcal{K}(H)$. Let $h := T(e)$. Then

$$U(T(e)) = \beta(T(e)\otimes e)(f) = \beta(T\circ(e\otimes e))(f) = \beta(T)\beta(e\otimes e)(f)$$

$$= \beta(T)(f \otimes f)(f) = \sigma(p i_{E_1}(g \otimes f)p)(f) = \sigma(i_{E_1}(g \otimes f))(f)$$
$$= (g \otimes f)(f) = \langle f, f \rangle g = g.$$

Thus $g \in \operatorname{im} U$ and $\operatorname{im} P \subseteq \operatorname{im} U$.

Now if H is a finite-dimensional space, it follows that $\operatorname{im} U$ is finite-dimensional. Since $\operatorname{im} P \subseteq \operatorname{im} U$, this implies that P has finite rank and hence $\pi(P) = 0$. Now if $x \in C^*(G)$, then since π_{E_2} is surjective there exists an element $a \in C^*(E_2)$ for which $\pi_{E_2}(a) = x$. Since $\pi_{E_1}(\phi(a)) = \pi_{E_2}(a) = x$, it follows that $\tau_1(x) = \pi(\sigma(\phi(a)))$. But since $\phi(a) \in \operatorname{im} \phi = pC^*(E_1)p$ we have that $\phi(a) = p\phi(a)$ and thus $\tau_1(x) = \pi(\overline{\sigma}(p)\sigma(\phi(p)) = 0$. Since x was arbitrary this implies that $\tau_1 = 0$. Furthermore, since H is finite-dimensional, the extension associated to E_2 is $\tau_2 = 0$. Thus τ_1 and τ_2 are CK-equivalent.

Therefore, all that remains is to consider the case when H is infinite-dimensional. In this case $H = \mathcal{H}$ and $\mathcal{K}(H) = \mathcal{K}$. Furthermore, if S is any element of $\mathcal{K}$, then for all $h \in \mathcal{H}$ we have that

$$(\beta(S) \circ U)(h) = \beta(S)(\beta(h \otimes e)(f)) = \beta(Sh \otimes e)(f) = U(Sh).$$

Since U is an isometry this implies that $U^*\beta(S)U = S$ for all $S \in \mathcal{K}$. Therefore, $\operatorname{Ad}(U^*) \circ \beta$ is the inclusion map $i : \mathcal{K} \to \mathcal{B}$. Since $\operatorname{Ad}(U^*) \circ \beta = \operatorname{Ad}(U^*) \circ \sigma \circ \phi \circ i_{E_2}$, this implies that $\operatorname{Ad}(U^*) \circ \sigma \circ \phi$ is the unique map which makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "i_{E_2}"] \arrow[dr, "i"'] & C^*(E_2) \arrow[d, "\operatorname{Ad}(U^*)\circ\sigma\circ\phi"] \\
& \mathcal{B}
\end{tikzcd}$$

Therefore, if τ_2 is (the Busby invariant of) the extension associated to $C^*(E_2)$, then by definition τ_2 is equal to the following. For any $x \in C^*(G)$ choose an $a \in C^*(E_2)$ for which $\pi_{E_2}(a) = x$. Then $\tau_2(x) := \pi(\mathrm{Ad}(U^*) \circ \sigma \circ \phi(a))$. Using the commutativity of diagram 5.3, this implies that

$$\begin{aligned}
\tau_2(x) =& \ \mathrm{Ad}(\pi(U^*)) \circ \pi(\sigma(\phi(a))) = \mathrm{Ad}(\pi(U^*)) \circ \tau_1(\pi_{E_1}(\phi(a))) \\
=& \ \mathrm{Ad}(\pi(U^*)) \circ \tau_1(\pi_{E_2}(a)) = \mathrm{Ad}(\pi(U^*)) \circ \tau_1(x).
\end{aligned}$$

So for all $x \in C^*(G)$ we have that

$$\tau_2(x) = \pi(U^*)\tau_1(x)\pi(U). \tag{5.4}$$

Now if a is any element of $C^*(E_2)$, then $\phi(a) \in pC^*(E_1)p$. Thus $\phi(a) = p\phi(a)$ and

$$\sigma(\phi(a)) = \sigma(p\phi(a)) = \overline{\sigma}(p)\sigma(\phi(a)) = P\sigma(\phi(a)).$$

Hence $\operatorname{im}\sigma(\phi(a)) \subseteq \operatorname{im} P \subseteq \operatorname{im} U$, and we have that

$$UU^*\sigma\phi(a) = \sigma\phi(a) \qquad \text{for all } a \in C^*(E_2).$$

Furthermore, for any $x \in C^*(G)$, we may choose an $a \in C^*(E_2)$ for which $\pi_{E_2}(a) = x$, and using the commutativity of diagram 5.3 we then have that

$$\begin{aligned}
UU^*\sigma\phi(a) &= \sigma\phi(a) \\
\pi(UU^*)\pi\sigma\phi(a) &= \pi\sigma\phi(a) \\
\pi(UU^*)\tau_1\pi_{E_1}\phi(a) &= \tau_1\pi_{E_1}\phi(a) \\
\pi(UU^*)\tau_1\pi_{E_2}(a) &= \tau_1\pi_{E_2}(a)
\end{aligned}$$

$$\pi(UU^*)\tau_1(x) = \tau_1(x).$$

In addition, this implies that for any $x \in C^*(G)$ we have that $\pi(UU^*)\tau_1(x^*) = \tau_1(x^*)$, and taking adjoints this gives that

$$\tau_1(x)\pi(UU^*) = \tau_1(x) \qquad \text{for all } x \in C^*(G).$$

Thus for all $x \in C^*(G)$ we have

$$\tau_1(x) = \pi(UU^*)\tau_1(x)\pi(UU^*) = \pi(U)\Big(\pi(U^*)\tau_1(x)\pi(U)\Big)\pi(U^*) = \pi(U)\tau_2(x)\pi(U^*).$$

This, combined with Eq.(5.4), implies that $\tau_1 = \mathrm{Ad}(\pi(U))\circ\tau_2$ and $\tau_2 = \mathrm{Ad}(\pi(U^*))\circ\tau_1$. Since U is an isometry, τ_1 and τ_2 are CK-equivalent. $\square$

5.2 $C^*(G)$-embeddability for essential 1-sink extensions

In the previous section it was shown that if E_1 and E_2 are two 1-sink extensions of G, then one of the $C^*(E_i)$'s can be $C^*(G)$-embedded into the other if and only if their associated extensions are CK-equivalent. While this gives a characterization of $C^*(G)$-embeddability, it is somewhat unsatisfying due to the fact that CK-equivalence of the Busby invariants is not an easily checkable condition. We shall use the Wojciech map defined in Chapter 4 to translate this result into a statement about the Wojciech vectors of E_1 and E_2. We shall do this for essential 1-sink extensions in this section, and in the next section we shall consider non-essential 1-sink extensions.

Theorem 5.2.1. *Let G be a row-finite graph which satisfies Condition (L). Also let*

E_1 and E_2 be essential 1-sink extensions of G. Then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if

$$[\omega_{E_1}] = [\omega_{E_2}] \quad \textit{in } \operatorname{coker}(A_G - I),$$

where ω_{E_i} is the Wojciech vector of E_i and $A_G - I : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$.

Proof. Let τ_1 and τ_2 be the extensions associated to E_1 and E_2, respectively. It follows from Corollary 5.1.3 that τ_1 and τ_2 are in the same equivalence class in $\operatorname{Ext}(C^*(G))$ if and only if one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other. Since E_1 and E_2 are essential 1-sink extensions of G, the graph G contains no sinks. The claim then follows from the fact that the Wojciech map $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ is an isomorphism that takes τ_i to the class $[\omega_{E_i}]$ in $\operatorname{coker}(A_G - I)$. $\square$

5.3 $C^*(G)$-embeddability for non-essential 1-sink extensions

Recall from §2.3 that if E is a 1-sink extension of G, then $\Lambda_E = \Lambda_G \cup \{\lambda_{v_0}\}$. Also recall that we define $\overline{v_0} := \bigcup\{\gamma : \gamma \text{ is a maximal tail in } G \text{ and } \gamma \geq v_0\}$. In this section we prove an analogue of Theorem 5.2.1 for non-essential extensions.

Lemma 5.3.1. *Let G be a graph which satisfies Condition (K), and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If $C^*(E_2)$ is $C^*(G)$-embeddable into $C^*(E_1)$, then $\overline{v_1} = \overline{v_2}$.*

Proof. Let $\phi : C^*(E_2) \to C^*(E_1)$ be a $C^*(G)$-embedding. Also let $p \in \mathcal{M}(C^*(E_1))$ be the projection which determines the full corner $\operatorname{im} \phi$. Now for $i \in \{1, 2\}$ we have that $\Lambda_{E_i} = \Lambda_G \cup \{\lambda_{v_i}\}$ is homeomorphic to $\operatorname{Prim} C^*(E_i)$ via the map $\lambda \mapsto I_{H_\lambda}$, where $H_\lambda := E_i^0\backslash\lambda$ by [4, Corollary 6.5]. Furthermore, since ϕ embeds $C^*(E_2)$ onto a full corner of $C^*(E_1)$ it follows that $C^*(E_2)$ is Morita equivalent to $C^*(E_1)$ and the Rieffel correspondence is a homeomorphism between $\operatorname{Prim} C^*(E_2)$ and $\operatorname{Prim} C^*(E_1)$, which in this case is given by $I \mapsto \phi^{-1}(pIp)$ [80, Proposition 3.24]. Composing the homeomorphisms which we have described, we obtain a homeomorphism from $h : \Lambda_{E_2} \to \Lambda_{E_1}$, where $h(\lambda)$ is the unique element of Λ_{E_1} for which $\phi(I_{H_\lambda}) = pI_{H_{h(\lambda)}}p$.

We shall now show that this homeomorphism h is equal to the map h described in Lemma 2.3.2; that is, h restricts to the identity on Λ_G. Let $\lambda \in \Lambda_G \subseteq \Lambda_{E_2}$. We begin by showing that $h(\lambda) \in \Lambda_G$. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E_2-family, and let $\{t_f, q_w\}$ be the canonical Cuntz-Krieger E_1-family. Since $\lambda \in \Lambda_G$ it follows that $v_2 \notin \lambda$. Therefore $v_2 \in H_\lambda$ and $p_{v_2} \in I_{H_\lambda}$. Consequently, $\phi(p_{v_2}) \in \phi(I_{H_\lambda})$, and since $\phi(p_{v_2}) = q_{v_1}$ and $pI_{H_{h(\lambda)}}p = \phi(I_{H_\lambda})$ it follows that $q_{v_1} \in pI_{H_{h(\lambda)}}p \subseteq I_{H_{h(\lambda)}}$. Thus $v_1 \in H_{h(\lambda)}$ and $v_1 \notin h(\lambda)$. It follows that $h(\lambda) \neq \lambda_{v_1}$, and hence $h(\lambda) \in \Lambda_G$.

We shall now proceed to show that $h(\lambda) = \lambda$. Since $h(\lambda) \in \Lambda_G$ it follows that $H_{v_1} \subseteq H_{h(\lambda)}$. Thus $\ker \pi_{E_1} = I_{H_{v_1}} \subseteq I_{H_{h(\lambda)}}$. Now let $w \in \lambda$. If we let $\{u_g, r_x\}$ be the canonical Cuntz-Krieger G-family, then since $w \in G^0$ we have that $\pi_{E_2}(p_w) = r_w$. It then follows that

$$\pi_{E_1}(\phi(p_w) - q_w) = \pi_{E_1}(\phi(p_w)) - \pi_{E_1}(q_w) = \pi_{E_2}(p_w) - \pi_{E_1}(q_w) = r_w - r_w = 0.$$

Thus $\phi(p_w) - q_w \in \ker \pi_{E_1} \subseteq I_{H_{h(\lambda)}}$. We shall now show that $q_w \notin I_{H_{h(\lambda)}}$. To do this we suppose that $q_w \in I_{H_{h(\lambda)}}$ and arrive at a contradiction. If $q_w \in I_{H_{h(\lambda)}}$, then we would have that $\phi(p_w) \in I_{H_{h(\lambda)}}$. Thus $p\phi(p_w)p \in pI_{H_{h(\lambda)}}p$ and $p\phi(p_w)p \in \phi(I_{H_\lambda})$. Now $\phi(p_w) \in \phi(C^*(E_2))$ and $\phi(C^*(E_2)) = pC^*(E_1)p$. Hence $p\phi(p_w)p = \phi(p_w)$ and we have that $\phi(p_w) \in \phi(I_{H_\lambda})$. Since ϕ is injective this implies that $q_w \in I_{H_\lambda}$ and $w \in H_\lambda$ and $w \notin \lambda$ which is a contradiction. Therefore we must have that $q_w \notin I_{H_{h(\lambda)}}$ and

$w \notin H_{h(\lambda)}$ and $w \in h(\lambda)$. Hence $\lambda \subseteq h(\lambda)$.

To show inclusion in the other direction let $w \in h(\lambda)$. Then $w \in H_{h(\lambda)}$ and $q_w \notin I_{H_{h(\lambda)}}$. As above, it is the case that $\phi(p_w) - q_w \in I_{H_{h(\lambda)}}$. Therefore, $\phi(p_w) \notin I_{H_{h(\lambda)}}$ and since $pI_{H_{h(\lambda)}}p \subseteq I_{H_{h(\lambda)}}$ it follows that $\phi(p_w) \notin pI_{H_{h(\lambda)}}p$ or $\phi(p_w) \notin \phi(I_{H_\lambda})$. Thus $p_w \notin I_{H_\lambda}$ and $w \notin H_\lambda$ and $w \in \lambda$. Hence $h(\lambda) \subseteq \lambda$.

Thus $\lambda = h(\lambda)$ for any $\lambda \in \Lambda_G$, and the map $h : \Lambda_{E_2} \to \Lambda_{E_1}$ restricts to the identity on Λ_G. Since this map is a bijection it must therefore take λ_{v_2} to λ_{v_1}. Therefore h is precisely the map described in Lemma 2.3.2, and it follows from Lemma 2.3.2 that $\overline{v_1} = \overline{v_2}$. ☐

Definition 5.3.2. Let G be a row-finite graph which satisfies Condition (K). If (E, v_0) is a 1-sink extension of G we define

$$H_E := G^0 \backslash \overline{v_0}.$$

We call H_E the *inessential part of E*.

Lemma 5.3.3. *Let G be a row-finite graph which satisfies Condition (K) and let (E, v_0) be a 1-sink extension of G. Then H_E is a saturated hereditary subset of G^0.*

Proof. Let $v \in H_E$ and $e \in G^1$ with $s(e) = v$. If $r(e) \notin H_E$, then $r(e) \in \overline{v_0}$ and hence $r(e) \in \gamma$ for some $\gamma \in \chi_G$ with the property that $\gamma \geq v_0$. Since maximal tails are backwards hereditary this implies that $v = s(e) \in \gamma$. Hence $v \in \overline{v_0}$ and $v \notin H_E$ which is a contradiction. Thus we must have $r(e) \in H_E$ and H_E is hereditary.

Suppose that $v \notin H_E$. Then $v \in \overline{v_0}$ and $v \in \gamma$ for some $\gamma \in \chi_G$ with the property that $\gamma \geq v_0$. Since maximal tails contain no sinks there exists an edge $e \in G^1$ with $s(e) = v$ and $r(e) \in \gamma$. Thus $r(e) \in \overline{v_0}$ and $r(e) \notin H_E$. Hence H_E is saturated. ☐

Remark 5.3.4. Recall that if A is a C^*-algebra, then there is a lattice structure on the set of ideals of A given by $I \wedge J := I \cap J$ and $I \vee J :=$ the smallest ideal containing $I \cup J$. Furthermore, if G is a graph then the set of saturated hereditary subsets of G^0 also has a lattice structure given by $H_1 \wedge H_2 := H_1 \cap H_2$ and $H_1 \vee H_2 :=$ the smallest saturated hereditary subset containing $H_1 \cup H_2$. If G is a row-finite graph satisfying Condition (K), then it is shown in [4, Theorem 4.1] that the map $H \mapsto I_H$, where I_H is the ideal in $C^*(G)$ generated by $\{p_v : v \in H\}$, is a lattice isomorphism from the lattice of saturated hereditary subsets of G^0 onto the lattice of ideals of $C^*(G)$. We shall make use of this isomorphism in the following lemmas in order to calculate $\ker \tau$ for an extension $\tau : C^*(G) \to \mathcal{Q}$.

Lemma 5.3.5. *Let $0 \to \mathcal{K} \overset{i_E}{\to} E \overset{\pi_E}{\to} A \to 0$ be a short exact sequence, and let σ and τ be the unique maps which make the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r, "i_E"] \arrow[d, equal] & E \arrow[r, "\pi_E"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & \mathcal{B} \arrow[r, "\pi"] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

commute. Then $\ker(\pi \circ \sigma) = i_E(\mathcal{K}) \vee \ker \sigma$ *and* $\ker \tau = \pi_E(i_E(\mathcal{K}) \vee \ker \sigma)$.

Proof. Since $\ker(\pi \circ \sigma)$ is an ideal which contains $i_E(K)$ and $\ker \sigma$, it follows that $i_E(\mathcal{K}) \vee \ker \sigma \subseteq \ker(\pi \circ \sigma)$.

Conversely, if $x \in \ker(\pi \circ \sigma)$ then $\pi(\sigma(x)) = 0$ and $\sigma(x) \in \mathcal{K} = \sigma(i_E(\mathcal{K}))$. Thus $\sigma(x) = \sigma(a)$ for some $a \in i_E(\mathcal{K})$. Hence $x - a \in \ker \sigma$ and $x \in i_E(\mathcal{K}) \vee \ker \sigma$. Thus $\ker(\pi \circ \sigma) = i_E(\mathcal{K}) \vee \ker \sigma$.

In addition, the commutativity of the above diagram implies that $\pi^{-1}(\ker \tau) = \ker(\pi \circ \tau)$. Since π_E is surjective it follows that $\ker \tau = \pi_E(\ker(\pi \circ \sigma))$ and from the previous paragraph $\ker \tau = \pi_E(i_E(\mathcal{K}) \vee \ker \sigma)$. $\square$

For Lemmas 5.3.6 and 5.3.7 fix a row-finite graph G which satisfies Condition (K). Also let (E, v_0) be a fixed 1-sink extension of G which has the property that $E^*(v_0) := \{\alpha \in E^* : r(\alpha) = v_0\}$ contains infinitely many elements. Then $I_{v_0} \cong \mathcal{K}$, and we may choose an isomorphism $i_E : \mathcal{K} \to I_{v_0}$ and let σ and τ be the (unique) maps which make the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r, "i_E"] \arrow[d, equal] & C^*(E) \arrow[r, "\pi_E"] \arrow[d, "\sigma"] & C^*(G) \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & \mathcal{B} \arrow[r, "\pi"] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

commute. In particular, note that τ is the extension associated to E.

Lemma 5.3.6. *If σ is as above, then $\ker \sigma = I_{H'}$ where $H' := \{v \in E^0 : v \not\geq v_0\}$.*

Proof. Since G satisfies Condition (K) and E is a 1-sink extension of G, it follows that E also satisfies Condition (K). Thus $\ker \sigma = I_H$ for some saturated hereditary subset $H \subseteq E^0$. Let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. Now because $\sigma(q_{v_0})$ is a rank 1 projection, it follows that $q_{v_0} \notin \ker \sigma = I_H$ and thus $v_0 \notin H$. Since H is hereditary this implies that for any $w \in H$ we must have $w \not\geq v_0$. Hence $H \subseteq H'$.

Now let $F := E/H$; that is, F is the graph given by $F^0 := E^0 \backslash H$ and $F^1 := \{e \in E^1 : r(e) \notin H\}$. Then by [4, Theorem 4.1] we see that $C^*(F) \cong C^*(E)/I_H = C^*(E)/\ker \sigma$. Thus we may factor σ as $\overline{\sigma} \circ p$ to get the commutative diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "i_E"] \arrow[d, equal] & C^*(E) \arrow[r, "p"] \arrow[d, "\sigma"] & C^*(F) \arrow[dl, "\overline{\sigma}"] \\ \mathcal{K} \arrow[r, "i"] & \mathcal{B} & \end{tikzcd}$$

where p is the standard projection and $\overline{\sigma}$ is the monomorphism induced by σ. From the commutativity of this diagram it follows that $p \circ i_E : \mathcal{K} \to C^*(F)$ is injective. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger F-family in $C^*(F)$. Also let I_{v_0} be the ideal in $C^*(E)$ generated by q_{v_0}, and let J_{v_0} be the ideal in $C^*(F)$ generated by p_{v_0}. Using [54, Corollary 2.2] and the fact that any path in E with range v_0 is also a path in F, we have that

$$
\begin{aligned}
p(i_E(\mathcal{K})) &= p(I_{v_0}) \\
&= p(\overline{\operatorname{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_0\}) \\
&= \overline{\operatorname{span}}\{p(t_\alpha t_\beta^*) : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_0\} \\
&= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta) = v_0\} \\
&= J_{v_0}.
\end{aligned}
$$

From the commutativity of the above diagram it follows that $\overline{\sigma}$ is the (unique) map which makes the diagram

$$
\begin{tikzcd}
\mathcal{K} \arrow[r, "p \circ i_E"] \arrow[dr, "i"'] & C^*(F) \arrow[d, "\overline{\sigma}"] \\
& \mathcal{B}
\end{tikzcd}
$$

commute. Since $\overline{\sigma}$ is injective, it follows from [102, Proposition 2.2.14] that $p(i_E(\mathcal{K})) = J_{v_0}$ is an essential ideal in $C^*(F)$.

Now suppose that there exists $w \in F^0$ with $w \not\geq v_0$ in F. Then for every $\alpha \in F^*$ with $r(\alpha) = v_0$ we must have that $s(\alpha) \neq w$. Hence $p_w s_\alpha = 0$. Since $J_{v_0} = \overline{\operatorname{span}}\{s_\alpha s_\beta : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta) = v_0\}$ it follows that $p_w J_{v_0} = 0$. Since $p_w \neq 0$ this would imply that J_{v_0} is not an essential ideal. Hence we must have that $w \not\geq v_0$ for all $w \in F^0$.

Furthermore, if $\alpha \in F^*$ is a path with $s(\alpha) = w$ and $r(\alpha) = v_0$, then $\alpha \in E^*$. So if $w \not\geq v_0$ in E, then we must have that $w \not\geq v_0$ in F. Consequently, if $w \in H'$, then

$w \not\geq v_0$ in E, and we cannot have $w \in F^0$ because there is a path in F from every element of F^0 to v_0, and hence a path in E from every element of F^0 to v_0. Thus $w \notin F^0 := E^0 \backslash H$, and $w \in H$. Hence $H' \subseteq H$. $\square$

Lemma 5.3.7. *Let G and (E, v_0) be as before. If H_E is the inessential part of E, $H' := \{v \in E^0 : v \not\geq v_0\}$, and $H_{v_0} := E^0 \backslash G^0$; then in E we have that*

$$H' \vee H_{v_0} = H_E \cup H_{v_0}.$$

Proof. We shall first show that $H_E \cup H_{v_0}$ is a saturated hereditary subset of E^0. To see that it is hereditary, let $v \in H_E \cup H_{v_0}$. If $e \in E^1$ with $s(e) = v$, then one of two things must occur. If $e \in G^1$, then $s(e) = v$ must be in G^0 and hence $v \in H_E$. Since we know from Lemma 5.3.3 that H_E is a saturated hereditary subset of G, it follows that $r(e) \in H_E \subseteq H_E \cup H_{v_0}$. On the other hand, if $e \notin G^1$, then $r(e) \notin G^0$, and hence $r(e) \in H_{v_0} \subseteq H_E \cup H_{v_0}$. Thus $H_E \cup H_{v_0}$ is hereditary.

To see that $H_E \cup H_{v_0}$ is saturated, let $v \notin H_E \cup H_{v_0}$. Then $v \in \overline{v_0}$ and $v \in \gamma$ for some $\gamma \in \chi_G$ with the property that $\gamma \geq v_0$. Since maximal tails contain no sinks, there exists $e \in G^1$ with $s(e) = v$ and $r(e) \in \gamma$. But then $r(e) \in \overline{v_0}$ and $r(e) \notin H_E$. Since $e \in G^1$ this implies that $r(e) \notin H_E \cup H_{v_0}$. Thus $H_E \cup H_{v_0}$ is saturated.

Now since $H' \subset H_E$ we see that $H_E \cup H_{v_0}$ is a saturated hereditary subset which contains $H' \cup H_{v_0}$. Thus $H' \vee H_{v_0} \subseteq H_E \cup H_{v_0}$.

Conversely, suppose that $v \in H_E \cup H_{v_0}$. If S is any saturated hereditary subset of E which contains $H' \cup H_{v_0}$, then for every vertex $w \notin S$ we know that w cannot be a sink, because if it were we would have $w \not\geq v_0$. Thus we may find an edge $e \in G^1$ with $s(e) = w$ and $r(e) \notin S$. Furthermore, since $H' \cup H_{v_0} \subseteq S$ we must also have that $r(e) \geq v_0$. Thus if $v \notin S$, we may produce an infinite path α in G with $s(\alpha) = v$ and $s(\alpha_i) \geq v_0$ for all $i \in \mathbb{N}$. If we let $\gamma := \{w \in G^0 : w \geq s(\alpha_i) \text{ for some } i \in \mathbb{N}\}$, then $\gamma \in \chi_G$ and $\gamma \geq v_0$. Hence $v \in \overline{v_0}$ and $v \notin H_E \cup H_{v_0}$ which is a contradiction. Thus we must have $v \in S$ for all saturated hereditary subsets S containing $H' \cup H_{v_0}$. Hence $v \in H' \vee H_{v_0}$ and $H_E \cup H_{v_0} \subseteq H' \vee H_{v_0}$. □

Lemma 5.3.8. *Let G be a row-finite graph which satisfies Condition (K). Also let (E, v_0) be a 1-sink extension of G. If τ is the extension associated to E, then*

$$\ker \tau = I_{H_E}.$$

Proof. Consider the following two cases.

CASE 1: The set $E^*(v_0)$ contains finitely many elements.

Then from the definition of the extension associated to E, we have that $\tau = 0$. However, if $E^*(v_0)$ has only finitely many elements then $\gamma \ngeq v_0$ for all $\gamma \in \chi_G$. Hence $H_E = G^0$ and $I_{H_E} = C^*(G)$.

CASE 2: The set $E^*(v_0)$ contains infinitely many elements.

Then $I_{v_0} \cong \mathcal{K}$, and from Lemma 5.3.5 we have that $\ker \tau = \pi_E(I_{v_0} \vee \ker \sigma)$. Also Lemma 5.3.6 implies that $\ker \sigma = I_{H'}$. Since $I_{v_0} = I_{H_{v_0}}$, we see that from Lemma 5.3.7 that $I_{v_0} \vee \ker \sigma = I_{H_{v_0}} \vee I_{H'} = I_{H_{v_0} \vee H'} = I_{H_E \cup H_{v_0}}$.

Now if we let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger G-family in $C^*(G)$ and $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$, then

$$\ker \tau = \pi_E(I_{H_E \cup H_{v_0}}) = \pi_E(\langle \{q_v : v \in H_E \cup H_{v_0}\} \rangle) = \langle \{p_v : v \in H_E\} \rangle = I_{H_E}.$$

□

Lemma 5.3.9. *Let G be a row-finite graph which satisfies Condition (K), and let*

(E, v_0) *be a 1-sink extension of* G*. If* $w \in H_E$*, then*

$$\#\{\alpha \in E^* : s(\alpha) = w \textit{ and } r(\alpha) = v_0\} < \infty.$$

Proof. Suppose that there were infinitely many such paths. Then since G is row-finite there must exist an edge $e_1 \in G^1$ with $s(e_1) = w$ and with the property that there exist infinitely many $\alpha \in E^*$ for which $s(\alpha) = r(e_1)$ and $r(\alpha) = v_0$. Likewise, there exists an edge $e_2 \in G^1$ with $s(e_2) = r(e_1)$ and with the property that there are infinitely many $\alpha \in E^*$ for which $s(\alpha) = r(e_2)$ and $r(\alpha) = v_0$. Continuing in this fashion we produce an infinite path $e_1e_2e_3\ldots$ with the property that $r(e_i) \geq v_0$ for all $i \in \mathbb{N}$. If we let $\gamma := \{v \in G^0 : v \geq s(e_i) \text{ for some } i \in \mathbb{N}\}$, then $\gamma \in \chi_G$ and $\gamma \geq v_0$. Since $w \in \gamma$, it follows that $w \in \overline{v_0}$ and $w \notin H_E := E^0 \backslash \overline{v_0}$, which is a contradiction. □

Definition 5.3.10. Let G be a row-finite graph which satisfies Condition (K), and let (E, v_0) be a 1-sink extension of G. Then $n_E \in \prod_{H_E} \mathbb{Z}$ is the vector whose entries are given by

$$n_E(v) = \#\{\alpha \in E^* : s(\alpha) = v \text{ and } r(\alpha) = v_0\} \qquad \text{for } v \in H_E.$$

Note that the previous Lemma shows that $n_E(v) < \infty$ for all $v \in H_E$.

Lemma 5.3.11. *Let* G *be a row-finite graph which satisfies Condition (K), and let* (E, v_0) *be a 1-sink extension of* G*. If* $v \in H_E$ *and* $n_E(v) > 0$*, then* $A_G(v, v) = 0$*; that is, there does not exist an edge* $e \in G^1$ *with* $s(e) = r(e) = v$*.*

Proof. If there was such an edge $e \in G^1$, then $\gamma = \{w \in G^0 : w \geq v\}$ would be a maximal tail and since $n_E(v) > 0$ it would follow that $\gamma \geq v_0$. Since $v \in \gamma$ this implies that $v \in \overline{v_0}$ which contradicts the fact that $v \in H_E := G^0 \backslash \overline{v_0}$. $\square$

Lemma 5.3.12. *Let G be a row-finite graph which satisfies Condition (K), and let (E, v_0) be a 1-sink extension of G. Also let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. If $e \in G^1$ and $r(e) \in H_E$, then*

$$\operatorname{rank} \sigma(t_e) = n_E(r(e)).$$

Proof. If $n_E(r(e)) = 0$, then $r(e) \not\geq v_0$ and by Lemma 5.3.6 we have $\sigma(q_{r(e)}) = 0$. Since $\sigma(t_e)$ is a partial isometry $\operatorname{rank} \sigma(t_e) = \operatorname{rank} \sigma(t_e^* t_e) = \operatorname{rank} \sigma(q_{r(e)}) = 0$. Therefore we need only consider the case when $n_E(r(e)) > 0$.

Let B_E^1 denote the boundary edges of E. Also let $k_e := \max\{|\alpha| : \alpha \in E^*, s(\alpha) = r(e), \text{ and } r(\alpha) \in B_E^1\}$. By Lemma 5.3.9 we see that k_e is finite. We shall prove the claim by induction on k_e.

Base Case: $k_e = 0$. Let $e_1, e_2, \ldots, e_n$ be the boundary edges of E which have source $r(e)$. Then it follows from Lemma 4.2.8 that for $1 \leq i \leq n$

$$\operatorname{rank} \sigma(t_{e_i}) = \#Z(r(e_i), v_0)$$

where $Z(r(e_i), v_0)$ is the set of paths from $r(e_i)$ to v_0. Also if $f \in G^1$ is an edge with $s(f) = r(e)$, then because $n_E(r(e)) > 0$ Lemma 5.3.11 implies that $r(f) \neq r(e)$. Furthermore, since $k_e = 0$ we must have that $r(f) \not\geq v_0$. Therefore, just as before we must have $\operatorname{rank} \sigma(t_f) = 0$. Now since the projections $\{t_f t_f^* : f \in E^1 \text{ and } s(f) = r(e)\}$ are mutually orthogonal, we see that

$$\operatorname{rank} \sigma(t_e) = \operatorname{rank} \sigma(t_e^* t_e)$$

$$
\begin{aligned}
&= \operatorname{rank} \sum_{\substack{f \in E^1 \\ s(f)=r(e)}} \sigma(t_f t_f^*) \\
&= \operatorname{rank}\sigma(t_{e_1}) + \ldots + \operatorname{rank}\sigma(t_{e_n}) + \sum_{\substack{f \in G^1 \\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f t_f^*) \\
&= \#Z(r(e_1), v_0) + \ldots + \#Z(r(e_n), v_0). \\
&= n_E(r(e)).
\end{aligned}
$$

Inductive Step: Assume that the claim holds for all edges f with $k_f \leq m$. We shall now show that the claim holds for edges $e \in G^1$ with $k_e = m+1$. Let $e_1, e_2, \ldots, e_n$ be the exits of E with source $r(e)$. As above we have that $\operatorname{rank}\sigma(t_{e_i}) = \#Z(r(e_i), v_0)$ for all $1 \leq i \leq n$. Now if $f \in G^1$ is any edge with $s(f) = r(e)$, then Lemma 5.3.11 implies that $r(f) \neq r(e)$. Thus we must have that $k_f \leq k_e - 1$, and by the induction hypothesis $\operatorname{rank}\sigma(t_f) = n_E(r(f))$. Furthermore, since the projections $\{t_f t_f^* : f \in E^1 \text{ and } s(f) = r(e)\}$ are mutually orthogonal, we see that

$$
\begin{aligned}
\operatorname{rank}\sigma(t_e) &= \operatorname{rank}\sigma(t_e^* t_e) \\
&= \operatorname{rank} \sum_{\substack{f \in E^1 \\ s(f)=r(e)}} \sigma(t_f t_f^*) \\
&= \operatorname{rank}\sigma(t_{e_1}) + \ldots + \operatorname{rank}\sigma(t_{e_n}) + \sum_{\substack{f \in G^1 \\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f t_f^*) \\
&= \#Z(r(e_1), v_0) + \ldots + \#Z(r(e_n), v_0) + \sum_{\substack{f \in G^1 \\ s(f)=r(e)}} n_E(r(f)) \\
&= n_E(r(e)). \quad \square
\end{aligned}
$$

Let G be a row-finite graph which satisfies Condition (K) and let (E, v_0) be a 1-sink extension of G. If $H_E := G^0 \backslash \overline{v_0}$ is the inessential part of E, then since H_E is a saturated hereditary subset of G we may form the graph $F := G/H_E$ given by

$F^0 := G^0 \backslash H_E$ and $F^1 := \{e \in G^1 : r(e) \notin H_E\}$. With respect to the decomposition $G^0 = \overline{v_0} \cup H_E$ the vertex matrix A_G of G will then have the form

$$A_G = \begin{pmatrix} A_F & X \\ 0 & C \end{pmatrix}$$

where A_F is the vertex matrix of the graph F.

Furthermore, if $\tau : C^*(G) \to \mathcal{Q}$ is the Busby invariant of the extension associated to E, then by Lemma 5.3.8 we know that $\ker \tau = I_{H_E}$. Hence $C^*(G)/\ker \tau \cong C^*(F)$ by [4, Theorem 4.1] and we may factor τ as $\overline{\tau} \circ p$

$$\begin{tikzcd} C^*(G) \arrow[r, "p"] \arrow[d, "\tau"'] & C^*(F) \arrow[dl, "\overline{\tau}"] \\ \mathcal{Q} & \end{tikzcd}$$

where p is the standard projection and $\overline{\tau}$ is the monomorphism induced by τ. Note that since $\overline{\tau}$ is injective it is an essential extension of $C^*(F)$. Furthermore, with respect to the decomposition $G^0 = \overline{v_0} \cup H_E$ the Wojciech vector of E will have the form $\omega_E = \left(\begin{smallmatrix} \omega_E^1 \\ \omega_E^2 \end{smallmatrix}\right)$.

Lemma 5.3.13. *If $d : \mathrm{Ext}(C^*(F)) \to \mathrm{coker}(B_F - I)$ is the Cuntz-Krieger map, then*

$$d(\overline{\tau}) = [x]$$

where $[x]$ denotes the class in $\mathrm{coker}(B_F - I)$ of the vector $x \in \prod_{F^1} \mathbb{Z}$ given by $x(e) := \omega_E^1(r(e)) + (Xn_E)(r(e))$ for all $e \in F^1$.

Proof. Notice that because of the way H_E was defined, F will have no sinks. Also note that the diagram

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "i_E"] \arrow[d, equal] & C^*(E) \arrow[r, "\pi_E"] \arrow[d, "\sigma"] & C^*(G) \arrow[r, "p"] \arrow[d, "\tau"] & C^*(F) \arrow[dl, "\overline{\tau}"] \\
\mathcal{K} \arrow[r, "i"] & \mathcal{B} \arrow[r, "\pi"] & \mathcal{Q} &
\end{tikzcd}$$

commutes. Let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. For each $e \in F^1$ let

$$H_e := \operatorname{im} \sigma(t_e t_e^*)$$

and for each $v \in F^0$ let

$$H_v := \bigoplus_{\substack{e \in F^1 \\ s(e)=v}} H_e.$$

Also for each $v \in F^0$ define P_v to be the projection onto H_v and for each $e \in F^1$ define S_e to be the partial isometry with initial space $H_{r(e)}$ and final space H_e. Then $\{S_e, P_v\}$ is a Cuntz-Krieger F-family in $\mathcal{B}$. If we let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger F-family in $C^*(F)$, then by the universal property of $C^*(F)$ there exists a homomorphism $\tilde{t} : C^*(F) \to \mathcal{B}$ such that $\tilde{t}(s_e) = S_e$ and $\tilde{t}(p_v) = P_v$. Let $t := \pi \circ \tilde{t}$. Since G satisfies Condition (K), it follows that the quotient $F := G/H_E$ also satisfies Condition (K). Because $t(p_v) \neq 0$ for all $v \in F$ this implies that $\ker t = 0$ and t is an essential extension of $C^*(F)$.

Because $\sigma(t_e)$ is a lift of $\overline{\tau}(s_e)$ for all $e \in F^1$ we see that $\operatorname{ind}_{E_e} \overline{\tau}(s_e) t(s_e)$ equals the Fredholm index of $\sigma(t_e)S_e^*$ in H_e. Since S_e^* is a partial isometry with initial space H_e and final space $H_{r(e)} \subseteq \operatorname{im} \sigma(q_{r(e)})$, and since $\sigma(t_e)$ is a partial isometry with initial space $\operatorname{im} \sigma(q_{r(e)})$ and final space H_e, it follows that

$$\dim(\ker(\sigma(t_e)S_e^*)) = 0.$$

Also, $\sigma(t_e^*)$ is a partial isometry with initial space H_e and final space $\operatorname{im}\sigma(q_{r(e)})$, and S_e is a partial isometry with initial space $H_{r(e)}$ and final space H_e. Because $q_{r(e)} = \sum_{\{f\in E^1 : s(f)=r(e)\}} t_f t_f^*$ we see that

$$\operatorname{im}\sigma(q_{r(e)}) = H_{r(e)} \oplus \bigoplus_{\substack{f\in E^1\setminus F^1\\ s(f)=r(e)}} \operatorname{im}\sigma(t_f t_f^*)$$

Thus

$$\dim(\ker(S_e\sigma(t_e^*))) = \sum_{\substack{f\in E^1\setminus F^1\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f t_f^*) = \sum_{\substack{f\in E^1\setminus F^1\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f).$$

Now if f is any boundary edge of E, then by Lemma 4.2.8 we have that $\operatorname{rank}\sigma(t_f) = \#Z((r(f), v_0)$ where $Z((r(f), v_0)$ is the set of paths in F from $r(f)$ to v_0. Also if $f \in G^1$ is an edge with $r(f) \in H_E$, then $\operatorname{rank}\sigma(t_f) = n_E(r(f))$ by Lemma 5.3.12. Therefore,

$$\begin{aligned}
\dim(\ker(S_e\sigma(t_e^*))) &= \sum_{\substack{f \text{ is a boundary edge}\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f) + \sum_{\substack{r(f)\in H_E\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f)\\
&= \sum_{\substack{f \text{ is a boundary edge}\\ s(f)=r(e)}} \#Z(r(f), v_0) + \sum_{\substack{r(f)\in H_E\\ s(f)=r(e)}} n_E(r(f))\\
&= \omega_E(r(e)) + \sum_{w\in H_E} X(r(e), w) n_E(w).
\end{aligned}$$

Thus

$$\begin{aligned}
d_{\overline{\tau},t}(e) &= -\operatorname{ind}_{E_e}\overline{\tau}(s_e)t(s_e^*)\\
&= \omega_E(r(e)) + \sum_{w\in H_E} X(r(e), w) n_E(w)\\
&= \omega_E^1(r(e)) + (Xn_E)(r(e)). \quad \square
\end{aligned}$$

Lemma 5.3.14. *If* $\omega : \mathrm{Ext}(C^*(F)) \to \mathrm{coker}(A_F - I)$ *is the Wojciech map, then*

$$\omega(\overline{\tau}) = [\omega_E^1 + X n_E]$$

where $[\omega_E^1 + X n_E]$ *denotes the class of the vector* $\omega_E^1 + X n_E$ *in* $\mathrm{coker}(A_F - I)$.

Proof. By definition $\omega := \overline{S_F} \circ d$. From Lemma 5.3.13 we see that $d(\overline{\tau}) = [x]$, where $x(e) = \omega_E^1(r(e)) + (X n_E)(r(e))$ for all $e \in F^1$. Therefore, $\omega(\overline{\tau})$ is equal to the class $[y]$ in $\mathrm{coker}(A_F - I)$ where $y \in \prod_{F^0} \mathbb{Z}$ is the vector given by $y := S_F(x)$. Hence for all $v \in F^0$ we have that

$$y(v) = (S_F(x))(v) = \sum_{\substack{e \in F^1 \\ s(e)=v}} x(e) = \sum_{\substack{e \in F^1 \\ s(e)=v}} \omega_E^1(r(e)) + (X n_E)(r(e))$$

and thus for all $v \in F^0$ we have that

$$\begin{aligned}
&y(v) - \left(\omega_E^1(v) + (X n_E)(v)\right) \\
&= \left(\sum_{\substack{e \in F^1 \\ s(e)=v}} \omega_E^1(r(e)) + (X n_E)(r(e)) \right) - \left(\omega_E^1(v) + (X n_E)(v)\right) \\
&= \left(\sum_{w \in F^0} A_F(v,w) \left(\omega_E^1(w) + (X n_E)(w)\right) \right) - \left(\omega_E^1(v) + (X n_E)(v)\right).
\end{aligned}$$

Hence $y - (\omega_E^1 + X n_E) = (A_F - I)(\omega_E^1 + X n_E)$, and $\omega(\tau) = [y] = [\omega_E^1 + X n_E]$ in $\mathrm{coker}(A_F - I)$. $\square$

Remark 5.3.15. Let G be a row-finite graph which satisfies Condition (K) and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If $\overline{v_1} = \overline{v_2}$, then we may let $H := H_{E_1} = H_{E_2}$ and form the graph $F := G/H$ given by $F^0 := G^0 \backslash H$ and $F^1 := \{e \in G^1 : r(e) \notin H\}$). Then with respect to the decomposition $G^0 = (G^0 \backslash H) \cup H$, the vertex matrix of G has the form

$$A_G = \begin{pmatrix} A_F & X \\ 0 & C \end{pmatrix}$$

where A_F is the vertex matrix of F. Also with respect to this decomposition, the Wojciech vectors of E_1 and E_2 have the form $\omega_{E_1} = \binom{\omega^1_{E_1}}{\omega^2_{E_1}}$ and $\omega_{E_2} = \binom{\omega^1_{E_2}}{\omega^2_{E_2}}$.

For $i \in \{1, 2\}$, let $n_{E_i} \in \prod_H \mathbb{Z}$ denote the vector given by $n_{E_i}(v) = \#\{\alpha \in E_i^* : s(\alpha) = v \text{ and } r(\alpha) = v_i\}$.

Theorem 5.3.16. *Let G be a row-finite graph which satisfies Condition (K), and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. Using the notation in Remark 5.3.15, we have that one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if*

1. $\overline{v_1} = \overline{v_2}$
2. $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ *in* $\operatorname{coker}(A_F - I)$.

Proof. It follows from Lemma 5.3.1 that if one of the $C^*(E_i)$'s is $C^*(G)$-embeddable in the other, then $\overline{v_1} = \overline{v_2}$. Thus we may let $H := H_{E_1} = H_{E_2}$ and form the graph $F := G/H$ as discussed in Remark 5.3.15.

If we let τ_1 and τ_2 be the Busby invariants of the extensions associated to E_1 and E_2, then it follows from Lemma 5.3.8 that $\ker \tau_1 = \ker \tau_2 = I_H$. Thus for each $i \in \{1, 2\}$, we may factor τ_i as $\tau_i = \overline{\tau}_i \circ p$

$$\begin{tikzcd} C^*(G) \arrow[r, "p"] \arrow[d, "\tau_i"'] & C^*(F) \arrow[dl, "\overline{\tau}_i"] \\ \mathcal{Q} & \end{tikzcd}$$

where p is the standard projection and $\overline{\tau}_i$ is the monomorphism induced by τ_i.

It then follows from Theorem 5.1.2 that one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if τ_1 and τ_2 are CK-equivalent. Since $\tau_i = \overline{\tau}_i \circ p$ we see that τ_1 and τ_2 are CK-equivalent if and only if $\overline{\tau}_1$ and $\overline{\tau}_2$ are CK-equivalent. Furthermore, since $\overline{\tau}_1$ and $\overline{\tau}_2$ are essential extensions we see from Corollary 5.1.3 that $\overline{\tau}_1$ and $\overline{\tau}_2$ are CK-equivalent if and only if $\overline{\tau}_1$ and $\overline{\tau}_2$ are equal in $\operatorname{Ext}(C^*(F))$. If $\omega : \operatorname{Ext}(C^*(F)) \to \operatorname{coker}(A_F - I)$ is the Wojciech map, then this will occur if and only if $\omega(\overline{\tau}_1) = \omega(\overline{\tau}_2)$, and by Lemma 5.3.14 we see that this happens if and only if $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ in $\operatorname{coker}(A_F - I)$. $\square$

Remark 5.3.17. Note that when E_1 and E_2 are both essential we have $\overline{v_1} = \overline{v_2} = G^0$ and $H = \emptyset$. In this case $F = G$, X is empty, and $\omega^1_{E_i} = \omega_{E_i}$ for $i = 1, 2$. Thus the result for essential extensions in Theorem 5.2.1 is a special case of the above theorem.

In addition, we see that the above theorem gives a method of determining $C^*(G)$-embeddability from basic calculations with data that can be easily read of from the graphs. To begin, the condition that $\overline{v_1} = \overline{v_2}$ can be checked simply by looking at E_1 and E_2. In addition, the set H, the matrices A_F and X, and the vectors $\omega^1_{E_i}$ and n_{E_i} for $i = 1, 2$ can easily be read off from the graphs G, E_1, and E_2. Finally, determining whether $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ in $\operatorname{coker}(A_F - I)$ amounts to ascertaining whether $(\omega^1_{E_1} - \omega^1_{E_2}) + (X(n_{E_1} - n_{E_2})) \in \operatorname{im}(A_F - I)$, a task which reduces to checking whether a system of linear equations has a solution.

We now mention an interesting consequence of the above theorem.

Definition 5.3.18. Let G be a row-finite graph which satisfies Condition (K), and let (E, v_0) be a 1-sink extension of G. We say that E is *totally inessential* if $\overline{v_0} = \emptyset$; that is, if $\{\gamma \in \chi_G : \gamma \geq v_0\} = \emptyset$.

Corollary 5.3.19. *Let G be a row-finite graph which satisfies Condition (K). If (E_1, v_1) and (E_2, v_2) are 1-sink extensions of G which are totally inessential, then* *one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.*

Proof. Using the notation established in Remark 5.3.15 and the proof of Theorem 5.3.16, we see that if E_1 and E_2 are totally inessential, then $H = G^0$. Hence $F = \emptyset$ and $\overline{\tau}_1 = \overline{\tau}_2 = 0$. Thus $\overline{\tau}_1$ and $\overline{\tau}_2$ are trivially CK-equivalent. Hence τ_1 and τ_2 are CK-equivalent and it follows from Theorem 5.1.2 that one of the $C^*(E_i)$'s can be $C^*(G)$-embedded into the other.

Alternatively, we see that if E_1 and E_2 are totally inessential, then $F = \emptyset$, and provided that we interpret the condition that $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ in $\operatorname{coker}(A_F - I)$ as being vacuously satisfied, the previous theorem implies that one of the $C^*(E_i)$'s can be $C^*(G)$-embedded into the other. $\square$

Remark 5.3.20. The case when E_1 and E_2 are both essential and the case when E_1 and E_2 are both totally inessential can be thought of as the degenerate cases of Theorem 5.3.16. The first occurs when $\overline{v_0} = G^0$ and $H = \emptyset$, and the second occurs when $\overline{v_0} = \emptyset$ and $H = G^0$.

Example 5.3.21. Let G be the graph

$$\cdots \xrightarrow{e_{-2}} v_{-1} \xrightarrow{e_{-1}} v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots$$

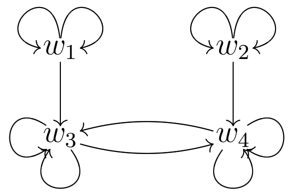
Note that $C^*(G) \cong \mathcal{K}$. Since G has precisely one maximal tail $\gamma := G^0$, we see that if E is any 1-sink extension of G, then E will either be essential or totally inessential. Furthermore, one can check that $A_G - I : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ is surjective. Thus if E_1 and E_2 are two essential 1-sink extensions of G, we will always have that $[\omega_{E_1}] = [\omega_{E_2}]$ in $\operatorname{coker}(A_G - I)$. In light of Theorem 5.2.1 and Corollary 5.3.19 we see that if E_1 and E_2 are two 1-sink extensions of G, then one of the $C^*(E_i)$'s can be $C^*(G)$-embedded in to the other if and only if they are both essential or both totally inessential.

Remark 5.3.22. Note that the statement of the result in Theorem 5.3.16 involves the $\omega^1_{E_i}$ terms from the Wojciech vectors, but does not make use of the $\omega^2_{E_i}$ terms. If for each $i \in \{1, 2\}$ we let $B^0_{E_i}$ denote the boundary vertices of E_i, then we see that the nonzero terms of $\omega^2_{E_i}$ are those entries which correspond to the elements of $B^0_{E_i} \cap H$. Furthermore, if $v \in B^0_{E_i} \cap H$ and there is a path from F^0 to v, then the value of $\omega^2_{E_i}(v)$ will affect the value of n_{E_i}. However, if there is no path from F^0 to v, then the value of $\omega^2_{E_i}(v)$ will be irrelevant to the value of n_{E_i}.

Therefore, whether one of the $C^*(E_i)$'s may be $C^*(G)$-embedded onto the other depends on two things: the number of boundary edges at vertices in F^0 (which determine the value of the $\omega^1_{E_i}$'s), and the number of boundary edges at vertices in H which can be reached by F^0 (which determine the value of the n_{E_i}'s). The boundary edges whose sources are elements of H that cannot be reached by F^0 will not matter.

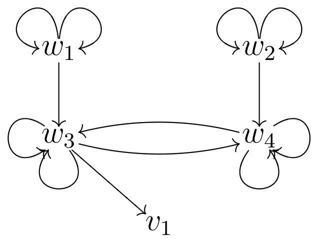
We end with an observation concerning $C^*(G)$-embeddability. In this chapter we have developed a fair number of results that tell us when the C^*-algebra of a 1-sink extension of G may be $C^*(G)$-embedded into the C^*-algebra of another 1-sink extension of G. Roughly speaking, this tells us when the C^*-algebra of one 1-sink extensions sits as a full corner in the C^*-algebra of the other in a way which preserves $C^*(G)$. In particular, it implies Morita equivalence of the C^*-algebras associated to the 1-sink extensions. Thus these results provide sufficient conditions for the C^*-algebras of the 1-sink extensions to be similar (i.e. Morita equivalent). It is natural to wonder how similar the C^*-algebras of two 1-sink extensions can be if neither can be $C^*(G)$-embedded into the other. It turns out that they can be very similar, in fact isomorphic, as the following example shows.

Example 5.3.23. Consider the following graph G

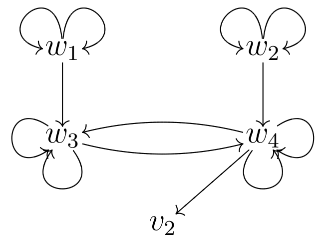


and its extensions E_1 and E_2;

E_1 :



E_2 :



One can see that the E_1 and E_2 are essential 1-sink extensions of G with Wojciech vectors $\omega_{E_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\omega_{E_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Furthermore, $A_G - I = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and one can check that $\omega_{E_1} - \omega_{E_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ is not an element of $\mathrm{im}(A_G - I)$. Since G satisfies Condition (L) we see from Theorem 5.2.1 that neither of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.

However, it is clear that $C^*(E_1) \cong C^*(E_2)$ since the graphs E_1 and E_2 are isomorphic. Thus we see that it is possible for two 1-sink extensions to have C^*-algebras that are very similar (in fact, isomorphic) without being able to $C^*(G)$-embed one into the other.

Appendix A

A primer on C^*-algebras of graphs

The intended purpose of this appendix is somewhat tangential to the main goals of this thesis. Here we present a survey of C^*-algebras associated to graphs. Because the main body of this thesis is fairly self-contained, only the very basics of what is presented here are necessary as background. Instead, the purpose of this appendix is to serve as a potential resource for future students who wish to learn about graph algebras. Therefore we endeavor to provide an overview of the important results and techniques of the theory, provide references to significant papers in the area, and summarize the current state of the study of graph algebras. It goes without saying that the choices of what to include here are influenced by the author's experiences, opinions, and tastes.

We assume that the reader is familiar with some elementary facts regarding C^*-algebras and also with the very basics of K-theory. Although we will provide almost no proofs here, we will try to give references to proofs of the main results. For the individual just beginning to study graph algebras, the author would suggest initially learning about C^*-algebras of row-finite graphs. One way to do this is to skim through the results in [55] and [54], and then perform a careful reading of [4]. Once familiar with row-finite graphs, those who are interested in C^*-algebras of arbitrary graphs would do well to consult [78], [29], [22], and [3]. In addition, one should be aware of the survey [51], which contains some nice examples and an excellent bibliography, and also the survey [64], which summarizes much of the early work on graph algebras.

A final word concerning groupoids. The study of C^*-algebras of graphs began with [55] and [54] where existence and basic facts regarding these objects were established. These initial treatments involved groupoid techniques, and due to technical requirements the graphs were assumed to be locally finite (i.e. each vertex emits and receives finitely many edges) and to have no sinks. Later it was shown that many of these results could be obtained by direct methods and for the most part groupoids could be avoided [4].

Although groupoids are still used by some authors (see [69], for example), they are much less pervasive than in the initial treatments. A student who is just entering the subject should be aware that one can go a long way in the study of graph algebras with only a little knowledge of groupoids (to be honest, the author knows only slightly more than the definition of a groupoid). Consequently, when first learning the theory one's time could best be used studying other things. In the discussions that follow we will make almost no mention of groupoids.

A.1 Graphs

A directed graph $G = (G^0, G^1, r, s)$ consists of a countable set G^0 of vertices, a countable set G^1 of edges, and functions $r, s : G^1 \to G^0$ that identify the range and source of each edge. A vertex that emits no edges is called a *sink* and a vertex that emits infinitely many edges is called an *infinite emitter*. A *singular vertex* is a vertex that is either a sink or an infinite emitter. We say that a graph is *row-finite* if each vertex emits only finitely many edges. (Note that a graph is row-finite if and only if it contains no infinite emitters.) We say that a graph is *locally finite* if each vertex both emits and receives finitely many edges.

A.1.1 Paths and loops in graphs

A *path* is a finite sequence of edges $\alpha := \alpha_1\alpha_2\dots\alpha_n$ for which $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$, and we say that such a path has length $|\alpha| = n$. For $n \geq 2$ we let G^n denote the set of all paths of length n, and set $G^* := \bigcup_{n\geq 0} G^n$. The maps r and s extend naturally to G^*. We also let G^∞ denote the set of infinite paths $\alpha := \alpha_1\alpha_2\dots$.

We define a relation on G^0 by setting $v \geq w$ if there exists a path $\alpha \in G^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Note that this relation is transitive, but it is not typically a partial order since we can have $v \leq w \leq v$ without having $v = w$.

A *loop* is a path whose range and source are equal, and for a given loop $\alpha := \alpha_1\alpha_2\dots\alpha_n$ we say that α is based at $s(\alpha_1) = r(\alpha_n)$. An *exit* for a loop α is an edge $e \in G^1$ for which $s(e) = s(\alpha_i)$ for some i but $e \neq \alpha_i$. A condition that is often imposed on graphs is the following.

Condition (L): Every loop in G has an exit; that is, for every loop $\alpha := \alpha_1\dots\alpha_n$ there exists $e \in G^1$ such that $s(e) = s(\alpha_i)$ for some i but $e \neq \alpha_i$.

We call a loop *simple* if it returns to its base point exactly once; that is $s(\alpha_1) \neq s(\alpha_i)$ for $2 \leq i \leq n$. Another important condition for graphs is stated here.

Condition (K): No vertex in G is the base of exactly one simple loop; that is, every vertex is either the base of no loops or the base of more than one simple loop.

Note that Condition (K) implies Condition (L).

A.1.2 Matrices associated to graphs

There are two important matrices that we shall associate to a graph G. The first is the *vertex matrix* A_G. We define A_G to be the (possibly infinite) $G^0 \times G^0$ matrix whose entries are given by

$$A_G(v, w) = \#\{e \in G^1 : s(e) = v \text{ and } r(e) = w\}.$$

For a row-finite graph it shall often be useful for us to view this matrix as the mapping $A_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ given by left multiplication. Here $\prod_{G^0} \mathbb{Z}$ is the direct product of $|G^0|$ copies of $\mathbb{Z}$. Since the graph G is row-finite, the entries of A_G are finite and the rows of A_G are eventually 0. Therefore left multiplication of the elements of $\prod_{G^0} \mathbb{Z}$ by A_G will involve the computation of a finite sum, and hence this multiplication is well defined. In addition, for a row-finite graph we can also view the matrix A_G^t as the mapping $A_G^t : \bigoplus_{G^0} \mathbb{Z} \to \bigoplus_{G^0} \mathbb{Z}$ given by left multiplication. Here $\bigoplus_{G^0} \mathbb{Z}$ is the direct sum of $|G^0|$ copies of $\mathbb{Z}$. Notice that the rows of A_G^t may contain infinitely many nonzero entries. However, since a vector in $\bigoplus_{G^0} \mathbb{Z}$ has entries that are eventually zero, multiplication by A_G^t will involve the computation of a finite sum and hence is also well defined. Furthermore, since G is row-finite it follows that the columns of A_G^t are eventually zero. Therefore when we multiply an element of $\bigoplus_{G^0} \mathbb{Z}$ by A_G^t we will get a vector with only finitely many nonzero terms, and thus A_G^t does in fact map into $\bigoplus_{G^0} \mathbb{Z}$.

The other matrix that we shall associate to a graph is the *edge matrix* B_G. We define B_G to be the (possibly infinite) $G^1 \times G^1$ matrix whose entries are given by

$$B_G(e,f) = \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

If G is a row-finite graph, then the rows of B_G will eventually be zero. Therefore, just as with the vertex matrix, when G is a row-finite graph we obtain mappings $B_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ and $B_G^t : \bigoplus_{G^0} \mathbb{Z} \to \bigoplus_{G^0} \mathbb{Z}$ given by left multiplication.

Throughout we shall use the symbol δ_v to denote the element of $\bigoplus_{G^0} \mathbb{Z}$ (and hence also $\prod_{G^0} \mathbb{Z}$) which has a 1 in the v^{th} entry and 0's elsewhere. We shall also say that a matrix is *row-finite* if its rows are eventually zero, and that a matrix is *column-finite* if its columns are eventually zero.

A.2 C^*-algebras associated to graphs

If G is a graph, a *Cuntz-Krieger G-family* in a C^*-algebra is a set of mutually orthogonal projections $\{p_v : v \in G^0\}$ and partial isometries $\{s_e : e \in G^1\}$ with mutually orthogonal ranges that satisfy the *Cuntz-Krieger relations*

1. $s_e^* s_e = p_{r(e)}$ for all $e \in G^1$
2. $p_v = \sum_{\{e \in G^1 : s(e) = v\}} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.
3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in G^1$.

Note that when G is row-finite Condition 2 implies Condition 3.

Definition A.2.1. If G is a graph, we let $C^*(G)$ denote the C^*-algebra generated by a universal Cuntz-Krieger G-family $\{s_e, p_v\}$. It is universal in the sense that if $\{t_e, q_v\}$

is a Cuntz-Krieger G-family in a C^*-algebra A, then there exists a homomorphism $\phi : C^*(G) \to A$ such that $\phi(s_e) = t_e$ and $\phi(p_v) = q_v$.

The existence of $C^*(G)$ is proven in [54, Theorem 2.1] for row-finite graphs, and in [29, Definition 1] for arbitrary graphs. It is easy to see that the universal property of $C^*(G)$ implies that it is unique. Also note that if $\{s_e, p_v\}$ is the generating Cuntz-Krieger family in $C^*(G)$, then the Cuntz-Krieger relations imply that $C^*(G)$ is generated by $\{s_e : e \in G^1\} \cup \{p_v : v \text{ is an infinite emitter}\}$. Hence when G is row-finite $C^*(G)$ is generated by $\{s_e : e \in G^1\}$.

For a path $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$, we define $s_\alpha := s_{\alpha_1}s_{\alpha_2} \ldots s_{\alpha_n}$. If G is a graph, then it is a consequence of the Cuntz-Krieger relations that words in $\{s_e, s_f^*\}_{e,f\in G^1}$ collapse to products of the form $s_\alpha s_\beta^*$ for $\alpha, \beta \in G^*$ satisfying $r(\alpha) = r(\beta)$. (If $\alpha \in G^0$, then α is a vertex of G and we interpret s_α as p_α.) In fact, using the Cuntz-Krieger relations one obtains the following formula

$$s_\beta^* s_\gamma = \begin{cases} s_{\gamma'} & \text{if } \gamma = \beta\gamma' \\ p_{r(\gamma)} & \text{if } \gamma = \beta \\ s_{\beta'}^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

(see [54, Lemma 1.1]). Therefore, because the family $\{s_\alpha s_\beta^*\}$ is closed under multiplication and involution, it follows that for any graph G

$$C^*(G) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in G^* \text{ and } r(\alpha) = r(\beta)\}. \tag{A.1}$$

Remark A.2.2. One should notice that the only commutative graph algebras are those associated to disjoint unions of the following graphs

 and 

The C^*-algebras of the above graphs are $\mathbb{C}$ and $C(\mathbb{T})$ respectively, and thus all commutative graph algebras will be direct sums of the algebras $\mathbb{C}$ and $C(\mathbb{T})$.

To see that these are the only graphs whose associated C^*-algebras are commutative, begin by letting G be a connected graph. Now if $C^*(G)$ is commutative, then for any edge e we see that Condition 1 and Condition 3 imply that $p_{r(e)} = s_e^* s_e = s_e s_e^* \leq p_{s(e)}$. Since the p_v's are mutually orthogonal this implies that $r(e) = s(e)$, and every edge must begin and end at the same vertex. Since G is connected this implies that G has only one vertex. Now suppose that e and f are edges beginning and ending at this vertex. Then $s_e s_e^* = s_e^* s_e = p_{r(e)} = p_{r(f)} = s_f^* s_f = s_f s_f^*$ and since these partial isometries have mutually orthogonal ranges this implies that $e = f$. Hence G is a graph with a single vertex and at most one edge.

Remark A.2.3. The Cuntz-Krieger algebras of [15] are naturally associated to topological Markov chains and thereby related to the subject of dynamical systems, particularly symbolic dynamics. In a similar way C^*-algebras of graphs are related to symbolic dynamics, and many results from this field have applications in the graph algebra setting. For example, such topics as shifts of finite type, shift equivalence, strong shift equivalence, and flow equivalence may all be interpreted for graph algebras. We will not address these issues here, but we mention some sources for the interested reader. A good introduction to the subject of symbolic dynamics is [58]. A discussion of flow equivalence in the graph algebra setting appears in [20], which is an article version of D. Drinen's thesis [19]. In addition, there are many graph operations from symbolic dynamics that preserve either isomorphism or Morita equivalence of the associated C^*-algebra. For C^*-algebras of row-finite graphs, many of these operations are discussed in [20]; e.g., outsplittings [20, §4.1], insplittings [20, §4.2], and delays [20, §3].

A.3 Representing graph algebras on Hilbert space

A convenient way to visualize graph algebras is to consider them as algebras of operators on a Hilbert space. Suppose that $C^*(G)$ is a graph algebra. By the Gelfand-Naimark Theorem, we may choose a faithful nondegenerate representation $\pi : C^*(G) \to \mathcal{B}(\mathcal{H})$. For each vertex and edge of G let us define $P_v := \pi(p_v)$ and $S_e := \pi(s_e)$. For each $v \in G^0$ let $\mathcal{H}_v$ be the image of P_v. (Since π is nondegenerate we see that $\mathcal{H} = \bigoplus_{v \in G^0} \mathcal{H}_v$.) Now for each $e \in G^1$ we see that Condition 1 in the Cuntz-Krieger relations implies that the initial space of the partial isometry S_e will be $\mathcal{H}_{r(e)}$. We shall let $\mathcal{H}_e$ denote the final space of S_e. (Note that S_e maps in a direction "opposite" the edge e.)

Now since the projections p_v are all mutually orthogonal, we see that the subspaces $\{\mathcal{H}_v\}_{v \in G^0}$ are all mutually orthogonal. In addition, since the partial isometries s_e all have mutually orthogonal ranges, we see that the subspaces $\{\mathcal{H}_e\}_{e \in G^1}$ are all mutually orthogonal. Now Condition 3 of the Cuntz-Krieger relations implies that $\mathcal{H}_e \subseteq \mathcal{H}_{s(e)}$ for all $e \in G^1$. Furthermore, Condition 2 of the Cuntz-Krieger relations implies that

$$\mathcal{H}_v = \bigoplus_{s(e)=v} \mathcal{H}_e \qquad \text{whenever } 0 < |s^{-1}(v)| < \infty.$$

When v is an infinite emitter, then it will still be the case that $\bigoplus_{s(e)=v} \mathcal{H}_e \subseteq \mathcal{H}_v$, however, in general $\mathcal{H}_v$ can — and will — be much larger than $\bigoplus_{s(e)=v} \mathcal{H}_e$.

Thus we have the following description of $C^*(G)$: we may represent $C^*(G)$ as operators on a Hilbert space $\mathcal{H} = \bigoplus_{v \in G^0} \mathcal{H}_v$, where $C^*(G)$ is generated by the projections P_v onto $\mathcal{H}_v$ and partial isometries S_e with initial space $\mathcal{H}_{r(e)}$ and final space $\mathcal{H}_e \subseteq \mathcal{H}_{s(e)}$. Furthermore, $\bigoplus_{s(e)=v} \mathcal{H}_e \subseteq \mathcal{H}_v$ with

$$\mathcal{H}_v = \bigoplus_{s(e)=v} \mathcal{H}_e \qquad \text{whenever } 0 < |s^{-1}(v)| < \infty.$$

and

$$\mathcal{H}_v \ominus \bigoplus_{s(e)=v} \mathcal{H}_e \neq 0 \qquad \text{whenever } v \text{ is a singular vertex.}$$

This representation provides a useful model to keep in mind when thinking about graph algebras.

A.4 Units, loops, and AF-algebras

The following is proven in [54, Proposition 1.4].

Proposition A.4.1. *If G is a graph, then $C^*(G)$ is unital if and only if G contains finitely many vertices. In this case the unit is the finite sum $\sum_{v \in G^0} p_v$.*

If G contains an infinite number of vertices, then we can list them as $G^0 = \{v_1, v_2, \ldots\}$, and the sequence $\{\sum_{i=1}^n p_{v_i}\}_n$ will be a strictly increasing approximate unit for $C^*(G)$ consisting of projections.

The presence of loops greatly affects the structure of $C^*(G)$. Furthermore, the effect that a loop has on $C^*(G)$ depends greatly on whether or not the loop has an exit. Recall that an AF-algebra is a C^*-algebra that is the direct limit of finite-dimensional C^*-algebras (AF stands for Approximately Finite). The following result was proven in [54, Theorem 2.4] for locally finite graphs, and was extended to arbitrary graphs in [22, Corollary 2.13].

Theorem A.4.2. *A graph G has no loops if and only if $C^*(G)$ is an AF-algebra.*

Thus the absence of loops implies that $C^*(G)$ is close to being a finite-dimensional algebra. We shall see in Section A.7 that if G has sufficiently many loops, then $C^*(G)$ is purely infinite. These results are often summarized by saying "when G has no loops $C^*(G)$ is small, and when G has many loops $C^*(G)$ is large".

Now if G is a finite graph with no loops, then G must have at least one sink. Furthermore, $C^*(G)$ will be a finite-dimensional C^*-algebra and we have the following result from [54, Corollary 2.3].

Proposition A.4.3. *Suppose G is a finite graph with no loops and let $\{v_1, v_2, \ldots, v_k\}$ be the sinks of G. If we let $n(v_i) := \#\{\alpha \in G^* : r(\alpha) = v_i\}$, then*

$$C^*(G) \cong \bigoplus_{i=1}^{n} M_{n(v_i)}(\mathbb{C}).$$

Example A.4.4. If G is the graph

$$\begin{tikzcd} w \arrow[r] \arrow[dr] & x \arrow[r, bend left] \arrow[r, bend right] & v_1 \\ & v_2 & \end{tikzcd}$$

then there are 5 paths ending at v_1 and 2 paths ending at v_2, so $C^*(G) \cong M_5(\mathbb{C}) \oplus M_2(\mathbb{C})$. (Do not forget to count paths of length zero.)

We conclude this section by describing the effect that loops without exits have on $C^*(G)$. If G is the graph consisting of a single vertex v and a single edge e with $s(e) = r(e) = v$, then $C^*(G) \cong C(\mathbb{T})$. In addition, if G is a graph consisting of a single simple loop $\alpha_1 \ldots \alpha_n$, then $C^*(G) \cong M_n(C(\mathbb{T})) \cong M_n(\mathbb{C}) \otimes C(\mathbb{T})$ [40, Lemma 2.4].

Now if G is a graph that contains a loop without an exit, then the following proposition is a consequence of [54, Proposition 2.1].

Proposition A.4.5. *If $\alpha = \alpha_1 \dots \alpha_n$ is a loop in G with no exits, and H is the subgraph defined by $H^0 := \{s(\alpha_i)\}_{i=1}^n$ and $H^1 := \{\alpha_i\}_{i=1}^n$, then*

$$I_H := \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in G^* \text{ and } r(\alpha) = r(\beta) \in H^0\}$$

is an ideal of $C^*(G)$ that is Morita equivalent to $C^*(H) \cong M_n(C(\mathbb{T}))$ and hence is also Morita equivalent to $C(\mathbb{T})$.

Now since $C(\mathbb{T})$ contains uncountably many ideals — one for each closed subset of $\mathbb{T}$ — the fact that I_H is Morita equivalent to $C(\mathbb{T})$ implies that I_H contains uncountably many ideals. Since I_H is an ideal of $C^*(G)$, the ideals of I_H will also be ideals of $C^*(G)$. Hence $C^*(G)$ contains uncountably many ideals. Thus the presence of loops without exits implies that there are uncountably many ideals in $C^*(G)$, and for each loop without an exit there will be a subcollection of the ideals of $C^*(G)$ with a structure isomorphic to the ideal structure of $C(\mathbb{T})$. Thus, very roughly speaking, loops without exits in G create portions of $C^*(G)$ that are like $C(\mathbb{T})$.

These observations imply that it is possible for the ideal structure of $C^*(G)$ to be fairly complicated. However, in many instances we will be concerned with special kinds of ideals. In Section A.5.2 we shall discuss a canonical action of $\mathbb{T}$ on a graph algebra, called the gauge action, and in Section A.6.1 we shall discuss how one may describe the ideals of $C^*(G)$ that are invariant under this action.

A.5 Universality of the Cuntz-Krieger relations

For a finite $\{0,1\}$-matrix A the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined to be the C^*-algebra generated by partial isometries satisfying relations determined by A. In order for this to define a unique isomorphism class, Cuntz and Krieger assumed that the matrix A satisfied a nondegeneracy condition which they called Condition (I) [15, Proposition 2.10]. For graph algebras the need for an analogue of Condition (I) is avoided by instead requiring the graph algebra to satisfy a universal property. If G is a graph, then $C^*(G)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger G-family $\{s_e, p_v\}$, and hence by general nonsense it is unique up to isomorphism. However, despite this uniqueness, it is still important to know when an arbitrary collection of partial isometries and projections satisfying the Cuntz-Krieger relations for G will generate a C^*-algebra isomorphic to $C^*(G)$. To phrase the question precisely:

Question 1. *Let G be a graph and let B be a C^*-algebra containing a Cuntz-Krieger G-family $\{t_e, q_v\}$. If we let $C^*(\{t_e, q_v\})$ denote the C^*-subalgebra of B generated by $\{t_e, q_v\}$, then when will it be the case that $C^*(G) \cong C^*(\{t_e, q_v\})$?*

Clearly, a necessary condition is that the partial isometries and projections of $\{t_e, q_v\}$ be nonzero. However, this is by no means sufficient as the following example shows.

Example A.5.1. Let G be the graph consisting of a single vertex v and a single edge e beginning and ending at v. Then we have seen that $C^*(G) \cong C(\mathbb{T})$. Let $\{s_e, p_v\}$ denote the generating Cuntz-Krieger family for $C^*(G)$, let x be any point of $\mathbb{T}$, and let I_x denote the ideal $I_x := \{f \in C(\mathbb{T}) : f(x) = 0\}$. Then the the quotient $C^*(G)/I_x$ will not be isomorphic to $C^*(G)$. However, the elements $\{s_e + I_x, p_v + I_x\}$ will form a Cuntz-Krieger G-family in the quotient $C^*(G)/I_x$ consisting of nonzero elements.

Now because of the universal property of $C^*(G)$, we know that any Cuntz-Krieger family $\{t_e, q_v\}$ will induce a homomorphism $\phi : C^*(G) \to B$ such that $\phi(s_e) = t_e$ and $\phi(p_v) = q_v$. Therefore we see see that asking whether $C^*(\{t_e, q_v\})$ is isomorphic to $C^*(G)$ is equivalent to asking the following:

Question 2. *If $\phi : C^*(G) \to B$ is a homomorphism between C^*-algebras, then under what conditions will ϕ be injective?*

It is this question that we shall address in the following theorems.

A.5.1 The Cuntz-Krieger Uniqueness Theorem

The following result was proven for C^*-algebras of row-finite graphs in [4, Theorem 3.1] and extended to arbitrary graph algebras in [78, Theorem 1.5] and [22, Corollary 2.12].

Theorem A.5.2 (Cuntz-Krieger Uniqueness). *Let G be a directed graph in which every loop has an exit, and suppose that $\{S_e, P_v\}$ and $\{T_e, Q_v\}$ are two Cuntz-Krieger G-families in which all the projections P_v and Q_v are nonzero. Then there is an isomorphism $\phi : C^*(\{S_e, P_v\}) \to C^*(\{Q_v, T_e\})$ such that $\phi(S_e) = T_e$ and $\phi(P_v) = Q_v$ for all $e \in G^1$ and $v \in G^0$.*

Corollary A.5.3. *Suppose that G is a directed graph in which every loop has an exit and suppose that $\{s_e, p_v\}$ is the generating Cuntz-Krieger family for $C^*(G)$. If $\phi : C^*(G) \to B$ is a homomorphism between C^*-algebras and $\phi(p_v) \neq 0$ for all $v \in G^0$, then ϕ is injective.*

Corollary A.5.4. *Suppose that G is a directed graph in which every loop has an exit and suppose that $\{s_e, p_v\}$ is the generating Cuntz-Krieger family for $C^*(G)$. If I is a nonzero ideal in $C^*(G)$, then $p_v \in I$ for some $v \in G^0$.*

We mention that there is also a version of Theorem A.5.2 for Exel-Laca algebras [27, Theorem 3.1].

Remark A.5.5. If G is a finite graph with no sources or sinks and B_G is the edge matrix of G, then $\mathcal{O}_{B_G} \cong C^*(G)$ [29, Proposition 9]. Furthermore, G will satisfy

Condition (L) if and only if B_G satisfies Condition (I) [54, Lemma 3.3]. Thus we see that Theorem A.5.2 is a generalization of the original uniqueness result of Cuntz and Krieger.

A.5.2 The Gauge-Invariant Uniqueness Theorem

We now examine the question of uniqueness for graphs that do not necessarily satisfy Condition (L). Due to the universal property of $C^*(G)$ there is a natural action of $\mathbb{T}$ on $C^*(G)$ called the gauge action. Rather surprisingly, the existence of a gauge action for a C^*-algebra generated by a nonzero Cuntz-Krieger G-family is enough to ensure that this C^*-algebra is isomorphic to $C^*(G)$.

If $z \in \mathbb{T}$ then the family $\{zs_e, p_v\}$ is another Cuntz-Krieger G-family that generates $C^*(G)$, and by the universal property there exists a homomorphism $\gamma_z : C^*(G) \to C^*(G)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. Since the homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z it follows that $\gamma_z \in \operatorname{Aut} C^*(G)$, and an $\epsilon/3$ argument using (A.1) shows that γ is a strongly continuous action of $\mathbb{T}$ on $C^*(G)$. This action is called the *gauge action* of $C^*(G)$.

The following result was proven for finite graphs in [40, Theorem 2.3], for row-finite graphs in [4, Theorem 2.1], and for arbitrary graphs in [3, Theorem 2.1].

Theorem A.5.6 (Gauge-Invariant Uniqueness). *Let G be a directed graph, let $\{S_e, P_v\}$ be a Cuntz-Krieger G-family in $\mathcal{B}(\mathcal{H})$, and let $\pi : C^*(G) \to \mathcal{B}(\mathcal{H})$ be the representation such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. If each P_v is nonzero and there is a strongly continuous action β of $\mathbb{T}$ on $C^*(\{S_e, P_v\})$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$, then π is faithful.*

Corollary A.5.7. *Let G be a directed graph and $\phi : C^*(G) \to B$ a homomorphism of C^*-algebras. If $\phi(p_v) \neq 0$ for all $v \in G^0$ and there exists a strongly continuous action*

β of $\mathbb{T}$ on $\operatorname{im}\phi$ such that $\beta_z \circ \phi = \phi \circ \gamma_z$ for all $z \in \mathbb{T}$, then ϕ is faithful.

We say that an ideal I in $C^*(G)$ is *gauge-invariant* if $\gamma_z(a) \in I$ for all $a \in I$ and $z \in \mathbb{T}$. Note that if I is gauge-invariant, then the gauge action descends to an action on $C^*(G)/I$.

Corollary A.5.8. *Let G be a directed graph and let I be a nonzero gauge-invariant ideal of $C^*(G)$. Then $p_v \in I$ for some $v \in G^0$.*

We mention that there is also a version of Theorem A.5.6 for Exel-Laca algebras [78, Theorem 2.7] as well as for Cuntz-Pimsner algebras [30, Theorem 4.1].

A.5.3 The General Cuntz-Krieger Uniqueness Theorem

As useful as the Cuntz-Krieger Uniqueness Theorem and the Gauge-Invariant Uniqueness Theorem are, there arise situations in which they are no help (see [37] for some examples). To rectify this Szymański has given necessary and sufficient conditions for the injectivity of a homomorphism from a graph algebra to a C^*-algebra [94, Theorem 1.2] . This result contains both the Cuntz-Krieger Uniqueness Theorem and the Gauge-Invariant Uniqueness Theorem as special cases [94, Corollary 1.3 and Corollary 1.4].

Theorem A.5.9 (General Cuntz-Krieger Uniqueness). *Suppose that G is a directed graph and that $\phi : C^*(G) \to B$ is a homomorphism between C^*-algebras. Then ϕ is injective if and only if the following two conditions are satisfied:*

1. *$\phi(p_v) \neq 0$ for all $v \in G^0$*
2. *For each simple loop $e_1 \ldots e_n$ without exits, the spectrum of $\phi(s_{e_1} \ldots s_{e_n})$ contains the entire unit circle.*

A.6 Ideals in graph algebras

Recall that for $v, w \in G^0$ we write $v \geq w$ if there exists a path $\alpha \in G^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. For subsets $K, L \subseteq G^0$ we write $K \geq L$ to mean that for each $v \in K$ there exists $w \in L$ such that $v \geq w$.

A subset $H \subseteq G^0$ is called *hereditary* if every vertex in H feeds only into H; that is, if $v \geq w$ and $v \in H$ implies that $w \in H$. A hereditary set H is said to be *saturated* if every nonsingular vertex that feeds only into H is itself in H; that is, if $0 < |s^{-1}(v)| < \infty$ and $\{r(e) : s(e) = v\} \subset H$ implies that $v \in H$. The *saturation* of a hereditary subset H is the smallest saturated hereditary subset $\overline{H}$ of G^0 containing H. There is a natural lattice structure on the saturated hereditary subsets of G given by $H_1 \wedge H_2 := H_1 \cap H_2$ and $H_1 \vee H_2 := \overline{H_1 \cup H_2}$.

If $S \subseteq G^0$ there is an inductive method for constructing the smallest saturated hereditary subset containing S. Define $H_0(S) := \{v \in G^0 : w \geq v \text{ for some } w \in S\}$. Then $H_0(S)$ is the smallest hereditary subset containing S. If we let

$$H_i(S) := H_{i-1}(S) \cup \{v \in G^0 : 0 < |s^{-1}(v)| < \infty \text{ and } r(s^{-1}(v)) \subseteq H_{i-1}(S)\}$$

for $i \in \mathbb{N}$, then $H(S) := \bigcup_{i=1}^{\infty} H_i(S)$ is the smallest saturated hereditary subset containing S. This inductive description is often a convenient way to think of the saturation of a set. In particular it is the motivation for the definition of the saturation of an index $i \in I$ of a $\{0,1\}$-matrix $A = \{A(i,j)\}_{i,j \in I}$, which is used to give criteria for the simplicity of the Exel-Laca algebra $\mathcal{O}_A$ in [91]. It also arises in [3, Remark 3.1].

A.6.1 Gauge-invariant ideals

In general the ideal structure of graph algebras is fairly complicated. Therefore we shall initially restrict our attention to the gauge-invariant ideals; that is, those ideals I in $C^*(G)$ for which $\gamma_z(a) \in I$ for all $a \in I$ and $z \in \mathbb{T}$. We shall see that saturated hereditary subsets can be used to describe the structure of the gauge-invariant ideals in $C^*(G)$.

If G is a row-finite graph, then the gauge-invariant ideals of $C^*(G)$ can be described in a particularly nice way. In addition, the quotient of $C^*(G)$ by a gauge-invariant ideal can be realized as a graph algebra in a natural way. Given a subset $H \subseteq G^0$ we define I_H to be the ideal in $C^*(G)$ generated by $\{p_v : v \in H\}$. The following result is proven in [4, Theorem 4.1] and is a direct generalization of [55, Theorem 6.6]. (Note, however, that the description given here differs slightly from that in [14] and [40] where the ideals are determined by a preorder on the set of loops in G. This is because in infinite graphs one has to account for infinite tails as well as loops.)

Theorem A.6.1. *Let $G := (G^0, G^1, r, s)$ be a row-finite directed graph.*

1. *The map $H \mapsto I_H$ is an isomorphism from the lattice of saturated hereditary subsets of G^0 onto the lattice of gauge-invariant ideals of $C^*(G)$. (With inverse given by $I \mapsto \{v : p_v \in I\}$.)*

2. *Suppose H is a saturated hereditary subset of G^0. If $F^0 := G^0 \backslash H$, $F^1 := \{e \in G^1 : r(e) \notin H\}$, and $F = F(G \backslash H) := (F^0, F^1, r, s)$, then $C^*(G)/I_H$ is canonically isomorphic to $C^*(F)$ (i.e., if $\{s_e, p_v\}$ is the generating Cuntz-Krieger G-family for $C^*(G)$, then $\{s_e + I_H, p_v + I_H\}$ will be a generating Cuntz-Krieger F-family for $C^*(G)/I_H$).*

If the graph G is not row-finite, then the description of the gauge-invariant ideals of $C^*(G)$ becomes slightly more complicated. The reason for this is that if H is a saturated hereditary subset of G^0, then due to infinite emitters the set $\{s_e + I_H, p_v + I_H\}$ will not necessarily be a Cuntz-Krieger $F(G \backslash H)$-family.

For a saturated hereditary subset H of G^0 define

$$B_H := \{v \in G^0 : \ v \text{ is an infinite emitter and } 0 < |s^{-1}(v) \cap r^{-1}(G^0 \backslash H)| < \infty\}.$$

Now the set

$$\{(H, S) : \ H \text{ is a saturated hereditary subset of } G^0 \text{ and } S \subseteq B_H\} \tag{A.2}$$

will be a lattice as described in [22, §2] with $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. We also define

$$I_{(H,S)} := \text{the ideal in } C^*(G) \text{ generated by } \{p_v : v \in H\} \cup \{p^H_{v_0} : v_0 \in S\}$$

where

$$p^H_{v_0} := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e) \notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

The following result generalizes Theorem A.6.1 to arbitrary graphs. Part 1 was proven in [22, Theorem 3.5] for graphs satisfying Condition (K), and for general graphs in [3, Theorem 3.6]. Part 2 was proven in [3, Corollary 3.5]. In the statement of Theorem A.6.2 we use the notation of [22].

Theorem A.6.2. *Let $G := (G^0, G^1, r, s)$ be a directed graph.*

1. *The map $(H, S) \mapsto I_{(H,S)}$ is an isomorphism from the lattice described in (A.2)*

onto the lattice of gauge-invariant ideals of $C^*(G)$.

2. *Suppose H is a saturated hereditary subset of G^0 and $S \subseteq B_H$. We will define a graph $F = F_{(H,S)}$ as in Theorem A.6.1, except now we will need to add a new sink $\beta(v_0)$ for each $v_0 \in B_H \backslash S$ and extra edges $\beta(e)$ for each edge e with $r(e) = v_0$. Formally, we define*

$$F^0_{(H,S)} := (G^0 \backslash H) \cup \{\beta(v) : v \in B_H \backslash S\}$$

$$F^1_{(H,S)} := r^{-1}(G^0 \backslash H) \cup \{\beta(e) : e \in G^1 \text{ and } r(e) \in B_H \backslash S\}$$

and extend r, s by $s(\beta(e)) = s(e)$ and $r(\beta(e)) = \beta(v)$. Then $C^*(G)/I_{(H,S)}$ is canonically isomorphic to $C^*(F_{(H,S)})$.

A.6.2 Ideals and Condition (K)

In the proofs of Theorem A.6.1 and Theorem A.6.2 the Gauge-Invariant Uniqueness Theorem is invoked for the C^*-algebras associated to the graphs $F(G\backslash H)$ and $F_{(H,S)}$. If for every saturated hereditary subset H the graph $F(G\backslash H)$ and the graph $F_{(H,S)}$ satisfy Condition (L), then one may go through the proofs of Theorem A.6.1 and Theorem A.6.2 using the Cuntz-Krieger Uniqueness Theorem in place of the Gauge-Invariant Uniqueness Theorem. This allows one to deduce that every ideal of $C^*(G)$ has the form I_H or $I_{(H,S)}$, and hence every ideal is gauge-invariant.

Remark A.6.3. If G is a row-finite graph it is straightforward to show that $F(G\backslash H)$ satisfies Condition (L) for every saturated hereditary subset H if and only if G satisfies Condition (K). More generally, if G is an arbitrary graph, then $F_{(H,S)}$ satisfies Condition (L) for every saturated hereditary subset H and every $S \subseteq B_H$ if and only if G satisfies Condition (K).

From this remark we have the following result, which is discussed in [4, Theorem 4.4] and [4, Remark 4.5] for row-finite graphs, and in [3, Corollary 3.8] and [22, Theorem 3.5] for arbitrary graphs.

Theorem A.6.4. *If G is a graph that satisfies Condition (K), then all of the ideals of $C^*(G)$ are gauge-invariant. Consequently, Theorem A.6.1 and Theorem A.6.2 give a complete description of the ideals in $C^*(G)$.*

The description of ideals in terms of saturated hereditary sets is useful because in many cases one can easily read this information off from the graph.

Example A.6.5. If G is the graph

$$\begin{tikzcd} w \arrow[loop above] \arrow[loop below] & x \arrow[l] \arrow[r] & v \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

then we see that G satisfies Condition (K) and the saturated hereditary subsets of G^0 are $\{v, w, x\}$, $\{v\}$, $\{w\}$, and $\emptyset$. Thus the lattice of saturated hereditary subsets and the lattice of ideals in $C^*(G)$ are given by

$$\begin{tikzcd}[arrows=dash] & \{v,w,x\} \arrow[dl] \arrow[dr] & \\ \{v\} \arrow[dr] & & \{w\} \arrow[dl] \\ & \emptyset & \end{tikzcd} \qquad \begin{tikzcd}[arrows=dash] & I_{\{v,w,x\}} \arrow[dl] \arrow[dr] & \\ I_{\{v\}} \arrow[dr] & & I_{\{w\}} \arrow[dl] \\ & \{0\} & \end{tikzcd}$$

A program for the description of general ideals in C^*-algebras of graphs was began in [40] where results were obtained for finite graphs. In the introduction to [3] the authors promise that in a forthcoming sequel this program will be generalized to give a complete description of ideals in C^*-algebras of arbitrary graphs. A concise summary of these results is also forthcoming in [35].

A.7 Purely infinite C^*-algebras

A.7.1 Simple purely infinite C^*-algebras

J. Cuntz introduced in [12] what are now called the Cuntz algebras $\mathcal{O}_n$ (the universal C^*-algebra generated by n isometries whose range projections add up to the unit). He showed that these C^*-algebras have the property that for every nonzero x in $\mathcal{O}_n$ there exist $a, b \in \mathcal{O}_n$ such that $axb = 1$. Later he showed that for a simple C^*-algebra A this property is equivalent to having every nonzero hereditary C^*-subalgebra of A contain an infinite projection [13]. Cuntz named this property "purely infinite", and it has been found to be important in many aspects of C^*-algebra theory. Since then, the property of being simple and purely infinite has been reformulated in several other ways (see [87, Exercise 5.7], [57] and [103] for just a few).

Since the introduction of the Cuntz algebras the number of examples of simple purely infinite C^*-algebras has grown considerably. In particular, the Cuntz-Krieger algebras $\mathcal{O}_A$ corresponding to irreducible shifts of finite type have been shown to be purely infinite [15]. It was also shown in [7] that simple unital C^*-algebras with a certain nice property (approximate divisibility) are either stably finite or purely infinite. It remains an important open problem to decide if there exists a simple unital C^*-algebra which is neither stably finite nor purely infinite.

The notion of purely infinite has been extremely important in the classification problem for C^*-algebras. In 1989 George Elliott showed that a certain class of C^*-algebras (inductive limits of circle algebras of real rank zero) are classified up to isomorphism by their K-theory [25]. Elliott's paper started comprehensive research in what is now called the classification program of Elliott. Great strides have been made in dealing with the classification problem for purely infinite simple C^*-algebras. In addition to Elliott's work, numerous contributions were made by E. Kirchberg,

N. C. Phillips, and M. Rørdam [47, 48, 49, 84, 26]. These works culminated in the result that two stable, simple, purely infinite, separable, and nuclear C^*-algebras A and B are isomorphic if and only if they are KK-equivalent — a result attributed to both Kirchberg and Phillips independently, and depending on both Kirchberg's "Geneva Theorems" and the work of Rørdam and Elliott [26]. Throughout this work both the precise definition and the utility of the concept of purely infinite C^*-algebras has required the simplicity of the C^*-algebras under study.

A.7.2 Non-simple purely infinite C^*-algebras

Because of the usefulness of the concept of purely infinite C^*-algebras in the simple case, it is desirable to extend the definition of purely infinite to C^*-algebras which are not simple. In particular, this is a natural and necessary project if one wishes to provide a framework for an extension of the classification results obtained for purely infinite simple C^*-algebras.

Unfortunately, because of the many ways in which the property of being purely infinite can be formulated, it is unclear which of these formulations should be used to generalize the definition to non-simple C^*-algebras. Currently, there are two important definitions that have been proposed for non-simple purely infinite C^*-algebras. The first was proposed and used by authors such as Anantharaman-Delaroche [1]; Kumjian, Pask, and Raeburn [54]; and Laca and Spielberg [56]. However, very soon after this definition was proposed, Rørdam and Kirchberg rejected it because it would give examples violating the two main conditions that they believed purely infinite C^*-algebras should have: (a) a purely infinite C^*-algebra should not admit a nonzero trace, and (b) if B is any C^*-algebra, then $B \otimes \mathcal{O}_\infty$ should be purely infinite. In [50] Kirchberg and Rørdam proposed an alternate definition of purely infinite and gave arguments as to why they believed that their definition was the correct one to use. Of course, both of these definitions agree for simple C^*-algebras.

The question of which definition is the appropriate one is still open to debate, and each side has its advocates. Therefore, we shall examine both definitions in the context of graph algebras. In the next section we shall state both definitions, give necessary and sufficient conditions for graph algebras to satisfy them, and briefly compare the two notions.

A.7.3 Purely infinite graph algebras

Recall that if A is a C^*-algebra, then a C^*-subalgebra B of A is said to be *hereditary* if for $a \in A^+$ and $b \in B^+$ the inequality $a \leq b$ implies $a \in B$. Also recall that a projection in a C^*-algebra is said to be *infinite* if it is equivalent to a proper subprojection of itself.

The following definition has been used in [1], [56], [54], [27], and [4]:

Definition A.7.1 *(Purely Infinite — first definition).* A C^*-algebra A is *purely infinite* if every nonzero hereditary subalgebra of A contains an infinite projection.

We now give necessary and sufficient conditions for a graph algebra to be purely infinite in the sense of Definition A.7.1. This result was proven for locally finite graphs in [54, Theorem 3.9], for row-finite graphs in [4, Proposition 5.3], and for arbitrary graphs in [22, Corollary 2.14].

Theorem A.7.2. *Let G be a graph. Then $C^*(G)$ is purely infinite in the sense of Definition A.7.1 if and only if every vertex in G connects to a loop and every loop in G has an exit.*

The following definition is equivalent to the one given by Kirchberg and Rørdam in [50]:

Definition A.7.3 *(Purely Infinite — second definition).* A C^*-algebra A is *purely infinite* if every nonzero hereditary subalgebra of *every quotient* of A contains an infinite projection.

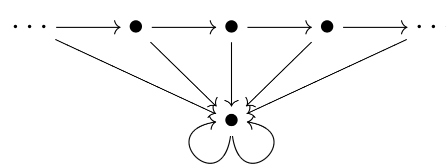
Note that the definition of purely infinite given in Definition A.7.1 is weaker than the definition of purely infinite given in Definition A.7.3, but that both definitions agree in the simple case. Also note that a C^*-algebra is purely infinite in the sense of Definition A.7.3 if and only if every quotient of it is purely infinite in the sense of Definition A.7.1.

Remark A.7.4. We mention that the formulation given in Definition A.7.3 is not the original definition of Kirchberg and Rørdam. In [50, Definition 4.1] Kirchberg and Rørdam define a C^*-algebra A to be purely infinite if there are no characters on A and if, for every pair of positive elements $a, b \in A$ such that a lies in the closed two-sided ideal generated by b, there exists a sequence $\{r_j\}_{j=1}^{\infty} \subseteq A$ with $r_j^* b r_j \to a$. Soon after they give this definition, however, they prove that it is equivalent to the one given in Definition A.7.3. (One direction is proven in [50, Proposition 4.7] and the converse is established in the discussion after [50, Question 4.8].)

Remark A.7.5. Kirchberg and Rørdam make convincing arguments in [50] to justify their definition of purely infinite. In particular, they verify that with their definition a purely infinite C^*-algebra will satisfy the two main conditions that they believed purely infinite C^*-algebras should have: (a) a purely infinite C^*-algebra does not admit a nonzero trace, and (b) if B is any C^*-algebra, then $B \otimes \mathcal{O}_\infty$ is purely infinite. In addition, they show that with their definition every ideal in a purely infinite C^*-algebra is purely infinite, every quotient of a purely infinite C^*-algebra is purely infinite, and the property of being purely infinite is preserved under extensions and Morita equivalence. They also establish that an approximately divisible C^*-algebra with no nonzero lower semi-continuous dimension function is purely infinite, and in particular, an approximately divisible exact C^*-algebra which admits no nonzero trace is purely infinite. One of their main points is that their definition of purely infinite provides a possible framework for extending the classification program to non-simple C^*-algebras.

In [38, Theorem 2.3] several conditions on a graph are given and shown to be equivalent to the associated C^*-algebra being purely infinite in the sense of Definition A.7.3. We shall conclude with an example that shows that the two notions of purely infinite are distinct for graph algebras.

Example A.7.6. Using Theorem A.7.2 we see that the C^*-algebra of the following graph is purely infinite in the sense Definition A.7.1.



However, if we I denote the ideal generated by the projection corresponding to the vertex based at the loops in the graph, then $C^*(G)/I$ will be isomorphic to the compact operators and hence is an AF-algebra. But then this quotient cannot contain an infinite projection, and therefore the C^*-algebra associated to the above graph is not purely infinite in the sense Definition A.7.3.

A.8 Simplicity of graph algebras

Cuntz and Krieger established criteria for the simplicity of the Cuntz-Krieger algebras in [15, Theorem 2.14]. Building on this work, conditions for simplicity of C^*-algebras of locally finite graphs were obtained in [55, Corollary 6.8] and similar results for row-finite graphs were obtained in [4, Proposition 5.1]. (However, the proof of [55, Corollary 6.8] is incomplete: the same direction is proved twice. The missing direction is proven in [4, Proposition 5.1] where the result is also extended to row-finite graphs.) In order to give the statement of this result we need a notion called cofinality.

Definition A.8.1. A graph G is said to be *cofinal* if for every infinite path $\alpha \in G^\infty$ and every vertex $v \in G^0$ there is a path from v to a vertex of α.

Theorem A.8.2. *Let G be a row-finite directed graph. Then $C^*(G)$ is simple if and only if the following conditions are satisfied:*

1. *G satisfies Condition (L)*
2. *G is cofinal.*

Remark A.8.3. Note that if a graph is cofinal and satisfies Condition (L), then it will also satisfy Condition (K). Therefore showing that the above conditions are sufficient for the simplicity of $C^*(G)$ is easy to prove: one simply shows that cofinality together with Condition (L) implies that G has no saturated hereditary subsets and then applies Theorem A.6.4.

Definition A.8.4. A graph G is said to *transitive* if for every $v, w \in G^0$ one has $v \geq w$.

Remark A.8.5. Note that if G is a row-finite graph that is transitive but not a single loop, then $C^*(G)$ is simple. However, there are many simple graph algebras $C^*(G)$ for which the graph G is not transitive.

Finding conditions for simplicity of general graph algebras and for Exel-Laca algebras has been an elusive goal of many authors in the past few years. It was not until recently that such conditions were obtained, and the preliminary work involved many partial results as well as high-powered techniques and sophisticated tools. When C^*-algebras of arbitrary (i.e. not necessarily row-finite) graphs were introduced in [29], it was shown that being transitive (without being a single loop) is a sufficient, but not necessary, condition for simplicity of the C^*-algebra [29, Theorem 3]. In [31, Corollary 4.5] it was shown that for graphs in which every vertex emits infinitely many edges, transitivity is both sufficient and necessary for simplicity of the C^*-algebra. In addition, Exel and Laca gave sufficient conditions for simplicity of the Exel-Laca algebras in [27, Theorem 14.1]. Necessary and sufficient conditions for simplicity of Exel-Laca algebras were finally obtained by Szymański in [92, Theorem 8] and his result could be adapted to give necessary and sufficient conditions for simplicity of C^*-algebras of arbitrary graphs [92, Theorem 12]. His conditions for the Exel-Laca algebras $\mathcal{O}_A$ were stated in terms of saturated hereditary subsets of the index set of A, and his conditions for graph algebras were stated in terms of saturated hereditary subsets of the graph's vertices. Shortly afterwards, independent results of [69, Theorem 4] and [22, Corollary 2.14] also gave necessary and sufficient conditions for simplicity of graph algebras in terms of reachability of certain vertices in the graph, thereby providing analogues to the statement of Theorem A.8.2.

We shall first state Szymański's simplicity criteria for Exel-Laca algebras [92, Theorem 8]. To do so we shall need the concept of the saturation of an index $i \in I$ for a $\{0,1\}$-matrix $\{A(i,j)\}_{i,j \in I}$.

Definition A.8.6. Suppose that I is the index set of a (possibly infinite) $\{0,1\}$-matrix $\{A(i,j)\}_{i,j \in I}$. For $i, j \in I$ let us write $j \geq i$ if there exists a finite sequence $i_0 = j, i_1, i_2, \ldots, i_n = i$ with $A(i_k, i_{k+1}) = 1$ for $0 \leq k \leq n-1$. For $j \in I$ we define $H_0(j) := \{j\} \cup \{i \in I : j \geq i\}$ and then inductively define $H_{n+1}(j)$ to be the union of $H_n(j)$ and the collection of $i \in I$ for which there exists a finite set $K \subseteq H_n(j)$ such that $A(i,t) \leq \max_{k \in K} A(k,t)$ for all $t \in I \backslash K$. The set $H(j) := \bigcup_{n=1}^{\infty} H_n(j)$ is called *the saturated hereditary subset of I containing j*.

Theorem A.8.7. *If $A = \{A(i,j)\}_{i,j\in I}$ is a $\{0,1\}$-matrix with no zero rows, then the Exel-Laca algebra $\mathcal{O}_A$ is simple if and only if the following two conditions are satisfied:*

1. *the graph $\mathrm{Gr}(A)$ satisfies Condition (L), where $\mathrm{Gr}(A)$ denotes the graph whose vertex set is I and has an edge from i to j if and only if $A(i,j) = 1$*

2. *$H(j) = I$ for all $j \in I$.*

Corollary A.8.8. *If G is a graph, then $C^*(G)$ is simple if and only if the following two conditions are satisfied:*

1. *G satisfies Condition (L)*

2. *the only saturated hereditary subsets of G are $\emptyset$ and G^0.*

The following statement of these conditions, which is analogous to the statement of Theorem A.8.2, appears in [69, Theorem 4] and [22, Corollary 2.14].

Theorem A.8.9. *Let G be a graph. Then $C^*(G)$ is simple if and only if the following conditions are satisfied:*

1. *G satisfies Condition (L)*

2. *G is cofinal*

3. *$G^0 \geq v_0$ for every singular vertex v_0.*

Again, we see that for an arbitrary graph G that is not a single loop, transitivity is a sufficient but not necessary condition for the simplicity of $C^*(G)$. However, Theorem A.8.9 shows that when every vertex is an infinite emitter, then transitivity also becomes necessary.

Corollary A.8.10. *If G is a graph in which every vertex emits infinitely many vertices, then $C^*(G)$ is simple if and only if G is transitive.*

We also see that Condition 3 in Theorem A.8.9 implies the following.

Corollary A.8.11. *If $C^*(G)$ is a simple graph algebra, then the graph G contains at most one sink.*

A.8.1 The dichotomy for simple graph algebras

With our characterization of simplicity for graph algebras, one can see that any simple graph algebra will be either AF or purely infinite. (Recall that for simple C^*-algebras the notions of purely infinite given in Definition A.7.1 and Definition A.7.3 coincide.) This fact is referred to as "the dichotomy for simple graph algebras". It was proven in [54, Corollary 3.11] for locally finite graphs with no sinks, extended to row-finite graphs in [4, Remark 5.6], and proven for arbitrary graph algebras in [91, Theorem 19] and [22, Remark 2.16].

Proposition A.8.12 (The Dichotomy). *Let G be a graph for which $C^*(G)$ simple. Then*

1. *$C^*(G)$ is an AF-algebra if G has no loops.*

2. *$C^*(G)$ is purely infinite if G contains a loop.*

Proof. Since $C^*(G)$ is simple, it follows from Theorem A.8.9 that G is cofinal and satisfies Condition (L). If G has no loops, then $C^*(G)$ is AF by Theorem A.4.2. If G has a loop, then every vertex connects to that loop due to cofinality, and $C^*(G)$ is purely infinite by Theorem A.7.2. □

Remark A.8.13. A simple C^*-algebra is said to be infinite if some matrix algebra over it contains an infinite projection; otherwise it is called stably finite. (The stably finite, simple C^*-algebras can again be divided into two subclasses depending on whether their stabilization contains a nonzero projection or not.) It is an open problem as to whether all infinite simple C^*-algebras are purely infinite. However, the dichotomy for simple graph algebras shows us that this is the case for graph algebras: if $C^*(G)$ is an infinite simple C^*-algebra, then it cannot be AF and hence it must be purely infinite.

A.9 Stability of graph algebras

Often in the study of C^*-algebras, and particularly in K-theory and KK-theory, one needs a C^*-algebra to have a sufficient amount of "room" within it. In this respect, a nice property for a C^*-algebra to have is that $A \cong M_2(A)$. In the absence of this property one may inquire as to whether $A \cong M_n(A)$ for some n. However, there is no reason to stop there. One may ask if A is isomorphic to "infinite matrices" with entries in A. Since the compact operators $\mathcal{K}$ are the closure of the finite rank operators, they provide a natural notion of "infinite matrices". Furthermore, since $M_n(A) \cong A \otimes M_n(\mathbb{C})$, we shall consider "infinite matrices with entries in A" to be $A \otimes \mathcal{K}$.

Definition A.9.1. We say that a C^*-algebra is *stable* if $A \cong A \otimes \mathcal{K}$. If A is a C^*-algebra, then the *stabilization of* A is defined to be $A \otimes \mathcal{K}$. We say that two C^*-algebras A and B are *stably isomorphic* if their stabilizations are isomorphic; that is, if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

Remark A.9.2. Note that since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, the stabilization of a C^*-algebra will be stable. Also the stabilization of a C^*-algebra is never unital.

Remark A.9.3. One often refers to a property of C^*-algebras as being stable if for any n the property holds for A if and only if the property holds for $M_n(A)$. Rørdam has shown in [85] that stability is not a stable property; in particular, he produces an example of a C^*-algebra A such that A is not stable but $M_2(A)$ is stable.

For a survey of known results regarding stabilization as well as various characterizations of stable C^*-algebras, one should consult Rørdam's excellent article [86].

The following result of Brown, Green, and Rieffel shows that stability is also an important property in the context of Morita equivalence.

Theorem A.9.4 (Brown-Green-Rieffel). *If two C^*-algebras are stably isomorphic, then they are Morita equivalent. If two σ-unital C^*-algebras are Morita equivalent, then they are stably isomorphic.*

This theorem shows that for σ-unital stable C^*-algebras Morita equivalence and isomorphism are the same. In particular, one should note that all separable C^*-algebras are σ-unital, and hence graph algebras are σ-unital. (In fact, any graph algebra has a countable approximate unit consisting of projections.)

The question of when a graph algebra will be stable has been addressed for locally finite graphs by Hjelmborg [34]. In order to discuss his results we need some terminology.

Definition A.9.5. Let G be a graph. A vertex $v \in G^0$ is said to be *left-infinite* if the set $L(v) := \{w \in G^0 : w \geq v\}$ is infinite. The vertex $v \in G^0$ is said to be *left-finite* if it is not left-infinite. A subset of vertices $S \subseteq G^0$ is said to be *left-infinite* (resp. *left-finite*) if every vertex in S is left-infinite (resp. left-finite).

The following result is proven in [34, Lemma 2.13]. Although it gives only a sufficient condition for stability, it is useful because this condition is often easy to verify.

Theorem A.9.6. *Let G be a locally finite graph. If G^0 is left infinite, then $C^*(G)$ is stable.*

The author believes that the above result will hold for row-finite graphs, and that the same proof will work. However, the author is uncertain as to whether the result holds for arbitrary graphs.

Recall that a *trace* on a C^*-algebra A is a positive linear map $\tau : A \to \mathbb{C}$ such that $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.

Definition A.9.7. If G is a row-finite graph, then a *graph trace* for G is a function $\tau_G : G^0 \to \mathbb{R}^+$ with the property that

$$\tau_G(v) := \sum_{\{e \in G^1 : s(e) = v\}} \tau_G(r(e)) \qquad \text{for all } v \in G^0.$$

We say that a graph trace is *bounded* if $\sum_{v \in G^0} \tau_G(v) < \infty$.

Remark A.9.8. If G is locally finite graph with no loops (so $C^*(G)$ is AF), then it is shown in [34, Lemma 2.8] that there is a one-to-one correspondence between bounded graph traces on G and bounded traces on $C^*(G)$. It is also conjectured that the result holds for more general graphs.

The following result appears in [34].

Theorem A.9.9. *Let G be a locally finite graph with no sinks. The following statements are equivalent:*

1. *$C^*(G)$ is stable*

2. *$C^*(G)$ admits no nonzero unital quotient and no nonzero bounded trace*

3. *If γ is any loop in G then the set $\{r(\gamma_i)\}_{i=1}^{|\gamma|}$ is left-finite, and the subset*

$$S := \{v \in G^0 : v \textit{ is a left-finite vertex which is on an infinite path}\}$$

   admits no nonzero bounded graph trace.

Again, the author believes that the above result will hold for row-finite graphs, and that the same proof will work.

A.10 The primitive ideal space of a graph algebra

In the analysis of commutative C^*-algebras one quickly becomes aware of the power of identifying nonzero homomorphisms with maximal ideals via the map $\phi \mapsto \ker \phi$ [70, 4.2.2]. This technique generalizes to noncommutative C^*-algebras, although one is now interested in kernels of irreducible representations.

Definition A.10.1. An ideal in a C^*-algebra A is called *primitive* if it is the kernel of an irreducible representation of A.

Definition A.10.2. An ideal I in a C^*-algebra A is said to be *prime* if whenever J and K are two ideals in A such that $J \cap K \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.

The following results are well known. Proofs may be found in [80, Proposition A.17(b)] and [80, Theorem A.49].

Theorem A.10.3. *Let A be a C^*-algebra. Every primitive ideal of A is prime. If A is separable, then every prime ideal of A is primitive.*

It is a rather surprising fact that general C^*-algebras may contain primitive ideals that are not prime [101]. However, since graph algebras are separable, the notions of primitive and prime will coincide for their ideals.

Definition A.10.4. If A is a C^*-algebra and F is a collection of primitive ideals of A, then we define the *closure* $\overline{F}$ of F to be

$$\overline{F} := \{P : P \text{ is a primitive ideal of } A \text{ and } \bigcap_{I \in F} I \subseteq P\}.$$

It turns out that this defines a topology on the set of primitive ideals of A with the closed sets being those collections of primitive ideals F for which $\overline{F} = F$. We call this topology the hull-kernel (or Jacobson) topology.

Definition A.10.5. The *primitive ideal space* $\operatorname{Prim} A$ of a C^*-algebra A is the set of primitive ideals of A endowed with the hull-kernel topology.

If G is a graph that satisfies Condition (K), then there is a nice description of $\operatorname{Prim} C^*(G)$. We shall discuss this result in the row-finite case first, and then discuss the general case. The following definition appears in [3] and [22].

Definition A.10.6. Let G be a graph. A nonempty subset $\gamma \subseteq G^0$ is called a *maximal tail* if it satisfies the following conditions:

(a) for every $w_1, w_2 \in \gamma$ there exists $z \in \gamma$ such that $w_1 \geq z$ and $w_2 \geq z$;

(b) for every $v \in \gamma$ that is not a singular vertex, there exists an edge e with $s(e) = v$ and $r(e) \in \gamma$;

(c) $v \geq w$ and $w \in \gamma$ imply $v \in \gamma$.

We let Λ_G denote the set of all maximal tails in G.

Remark A.10.7. Maximal tails were first defined for row-finite graphs in [4], and the definition was extended to arbitrary graphs in [3] and [22]. We mention a small discrepancy that exists between these definitions: part (b) of Definition A.10.6 does not agree with part (b) of the definition given in [4, Proposition 6.1]. In fact, if v_0 is a sink then the set $\lambda_{v_0} := \{v \in G^0 \mid v \geq v_0\}$ is a maximal tail according to Definition A.10.6, but λ_{v_0} is not considered to be a maximal tail according to [4, Proposition 6.1]. Furthermore, in [4] the set of all maximal tails was denoted χ_G, and Λ_G was defined to be $\Lambda_G := \chi_G \cup \{\lambda_{v_0} : v_0 \text{ is a sink}\}$. This set Λ_G was then used in [4] to describe $\operatorname{Prim} C^*(G)$. Although the definition of maximal tails appearing in

Definition A.10.6 differs from that in [4], we see that the definitions of the set Λ_G agree in both cases. Hence the object that we are truly interested in, namely Λ_G, is the same and the statement of most theorems in [4] will remain unchanged despite the slightly different terminology.

The following result was proven in [4, Corollary 6.5].

Theorem A.10.8. *Suppose G is a row-finite graph that satisfies Condition (K). Then there is a topology on Λ_G defined by*

$$\overline{S} = \{\delta \in \Lambda_E : \delta \geq \bigcup_{\lambda \in S} \lambda\}$$

for $S \subseteq \Lambda_G$, and with this topology the map $\lambda \mapsto I_{G^0 \setminus \lambda}$ is a homeomorphism from Λ_G onto $\operatorname{Prim} C^*(G)$.

In order to extend this description to general graphs satisfying Condition (K) we will need to consider not only maximal tails, but also special kinds of infinite emitters known as breaking vertices.

Definition A.10.9. If G is a graph, then a *breaking vertex* is an element $v \in G^0$ such that $|s^{-1}(v)| = \infty$ and $0 < |\{e \in G^1 : s(e) = v \text{ and } r(e) \geq v\}| < \infty$. We denote the set of breaking vertices of G by $BV(G)$.

We let $\Xi_G := \Lambda_G \cup BV(G)$ denote the disjoint union of the maximal tails and the breaking vertices. We shall see that the elements of Ξ_G correspond to the primitive ideals in $C^*(G)$. But first, we need to establish some notation to describe the topology on Ξ_G.

Definition A.10.10. Let G be a graph and let $S \subseteq G^0$. If γ is a maximal tail, then we write $\gamma \to S$ if $\gamma \geq S$. If v_0 is a breaking vertex in G, then we write $v_0 \to S$ if the set $\{e \in G^0 \mid s(e) = v_0, r(e) \geq S\}$ contains infinitely many elements.

Definition A.10.11. Let G be a graph that satisfies Condition (K). We define a map $\phi_G : \Xi_G \to \operatorname{Prim} C^*(G)$ as follows. For $\gamma \in \Lambda_G$ let $H(\gamma) := G^0 \setminus \gamma$ and define $\phi_G(\gamma) := I_{(H(\gamma), B_{H(\gamma)})}$. For $v_0 \in BV(G)$ we define $\phi_G(v_0) := I_{(H(\lambda_{v_0}), B_{H(\lambda_{v_0})} \setminus \{v_0\})}$.

The following result appears in [22, Theorem 4.10], and the correspondence (without the topology) was also described in [3, Corollary 4.8].

Theorem A.10.12. *Let G be a graph satisfying Condition (K). Then there is a topology on Ξ_G such that for $S \subseteq \Xi_G$,*

$$\overline{S} := \{\delta \in \Xi_G : \delta \to \bigcup_{\lambda \in S} \lambda\},$$

and the map ϕ_G given in Definition A.10.11 is a homeomorphism from Ξ_G onto $\operatorname{Prim} C^*(G)$.

When a graph does not satisfy Condition (K), it is still possible to identify the gauge-invariant primitive ideals. The following result is proven in [3, Theorem 4.7].

Definition A.10.13. If γ is a maximal tail in a graph G, then we say that *every loop in γ has an exit* if every loop in G with vertices in γ has an exit $e \in G^1$ with $r(e) \in \gamma$.

Theorem A.10.14. *Let G be a graph. Then the gauge-invariant primitive ideals in $C^*(G)$ are the ideals $I_{(H(\gamma), B_{H(\gamma)})}$ associated to maximal tails γ in which all loops have exits, and the ideals $I_{(H(\lambda_{v_0}), B_{H(\lambda_{v_0})} \setminus \{v_0\})}$ associated to breaking vertices $v_0 \in BV(G)$. Moreover, these ideals are distinct.*

A.11 K-theory

A.11.1 The classification program

In K-theory one associates to each C^*-algebra A two abelian groups $K_0(A)$ and $K_1(A)$. These two groups are countable if A is separable, and if A is unital the unit 1_A defines a class $[1_A]$ in $K_0(A)$. The importance of K-theory in this context is due in part to the fact that it respects many of the operations and techniques important to the study of C^*-algebras. If $A \times_\alpha G$ is a crossed product of A and G is equal to either $\mathbb{R}$ or $\mathbb{Z}$, then there are exact sequences relating the K-groups of $A \times_\alpha G$ and the K-groups of A. In addition, if A and B are Morita equivalent C^*-algebras, then $K_i(A) \cong K_i(B)$ for $i = 0, 1$. Consequently, the K-theory of a C^*-algebra is often computable, and the K-groups of many important classes of C^*-algebras are known.

The first triumph of K-theory for C^*-algebras occurred in Elliott's work on AF-algebras during the 1970's. If A is an AF-algebra, then $K_1(A) = 0$. Thus all of the K-theory information for an AF-algebra is contained in the group $K_0(A)$. Elliott showed that one can put an order structure on $K_0(A)$, and that two AF-algebras A and B are Morita equivalent if and only if $K_0(A)$ is order isomorphic to $K_0(B)$. In addition, if A and B are unital AF-algebras, then $A \cong B$ if and only if there is a scaled order isomorphism from $K_0(A)$ to $K_0(B)$ that takes $[1_A]$ to $[1_B]$. This remarkable result shows that AF-algebras are classified by their K-theory. Encouraged by this success Elliott conjectured that there might be a complete classification of all separable, nuclear C^*-algebras in terms of an invariant that has K-theory as an important ingredient. A great deal of effort has gone into examining special cases of this conjecture as well as attempting to ascertain what K-theory can tell us about various kinds of C^*-algebras. Currently, the efforts of many C^*-algebraists is focused on it, and the literature regarding this project comprises more than 100 articles.

Perhaps the greatest success of this K-theoretic approach is the Kirchberg-Phillips Classification Theorem, which was alluded to in Section A.7.1. This theorem is one of the crowning achievements in the modern theory of C^*-algebras, and was proven independently by Kirchberg and Phillips in the late 1990's. The result uses K-theory to classify C^*-algebras that are purely infinite, simple, separable, and nuclear and which satisfy the conditions of the Universal Coefficients Theorem. Although this seems like a plethora of conditions to place on our C^*-algebras, in practice one encounters many C^*-algebras that fall into this class. For brevity, we shall refer to a C^*-algebra that is purely infinite, simple, separable, and nuclear as a *Kirchberg algebra*.

In the summer of 1994 at a satellite conference to the International Congress of Mathematicians, Kirchberg announced three seminal theorems, which have come to be known as his "Geneva Theorems". During the fall of 1994, these theorems led Kirchberg and Phillips to independently obtain the following classification theorem: Two stable Kirchberg algebras A and B are isomorphic if and only if they are KK-equivalent; and moreover, every invertible element in $KK(A, B)$ lifts to an isomorphism from A to B. (A similar statement holds for unital Kirchberg algebras when one takes the position of the unit in K_0 into account.)

In order to apply this theorem to the Elliott conjecture one needs to know when two C^*-algebras are KK-equivalent. Rosenberg and Schochet showed in [88] that the Universal Coefficient Theorem (UCT) holds for all C^*-algebras A in a certain bootstrap class $\mathcal{N}$; and if the UCT holds for two C^*-algebras A and B, then A and B are KK-equivalent if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.

The Kirchberg-Phillips Theorem therefore implies that two unital (respectively, non-unital) Kirchberg algebras A and B in the bootstrap class $\mathcal{N}$ are isomorphic if and only if $(K_0(A), [1_A]) \cong (K_0(B), [1_B])$ and $K_1(A) \cong K_1(B)$ (respectively, $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$). Consequently Elliott's conjecture is verified in this important special case. We mention that it is also shown in [26] that for any pair (G_0, G_1) of countable abelian groups there is a Kirchberg algebra A in $\mathcal{N}$ such that $K_0(A) \cong G_0$ and $K_1(A) \cong G_1$.

The following is Phillips' version of the classification theorem and appears in [72, Theorem 4.2.4]. Kirchberg's version is not yet published, but a preliminary account, including proofs of his Geneva theorems and partial proofs of his version of the classification theorem, was circulated in 1994. Kirchberg's complete treatment is expected to appear soon in the form of a book.

Theorem A.11.1 (Kirchberg-Phillips Classification).

1. *Let A and B be purely infinite, simple, separable, unital, and nuclear C^*-algebras that satisfy the Universal Coefficients Theorem. If there are isomorphisms $\alpha_i : K_i(A) \to K_i(B)$ for $i = 0, 1$ with $\alpha_0([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ with $\phi_* = \alpha$.*

2. *Let A and B be purely infinite, simple, separable, nonunital and nuclear C^*-algebras which satisfy the Universal Coefficients Theorem. If there are isomorphisms $\alpha_i : K_i(A) \to K_i(B)$ for $i = 0, 1$, then there is an isomorphism $\phi : A \to B$ with $\phi_* = \alpha$.*

Corollary A.11.2. *Let A and B be purely infinite, simple, separable, unital, and nuclear C^*-algebras that satisfy the Universal Coefficients Theorem. If there are isomorphisms $\alpha_i : K_i(A) \to K_i(B)$ for $i = 0, 1$, then A and B are Morita equivalent.*

Proof. If A and B are purely infinite, simple, separable, and nuclear C^*-algebras which satisfy the Universal Coefficients Theorem, then it follows that their stabilizations $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ will have these properties also. If $K_i(A) \cong K_i(B)$ for $i = 0, 1$, then by the stability of K-theory we will have $K_i(A \otimes \mathcal{K}) \cong K_i(B \otimes \mathcal{K})$ for $i = 0, 1$.

Since the stabilization of a C^*-algebra is always nonunital, it follows from Part 2 of Theorem A.11.1 that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, and by Theorem A.9.4 the C^*-algebras A and B are Morita equivalent. $\square$

Remark A.11.3. It may seem unexpected that in the nonunital case of Theorem A.11.1 isomorphic K-theory implies not only Morita equivalence but also isomorphism of the C^*-algebras. However, this is understandable if one is aware of the following result of Zhang: every separable, nonunital, purely infinite C^*-algebra has the form $D \otimes \mathcal{K}$ for some unital purely infinite simple C^*-algebra D [104]. Thus nonunital Kirchberg algebras are stable, and since they are also separable Theorem A.9.4 implies that they are Morita equivalent if and only if they are isomorphic.

Remark A.11.4. It is an open problem as to whether the UCT holds for all nuclear C^*-algebras. A complete confirmation of Elliott's classification conjecture in the simple, infinite case depends on an affirmative solution to this open problem (at least for Kirchberg algebras) as well as an affirmative solution to the problem of whether all infinite, simple C^*-algebras are purely infinite.

A.11.2 K-theory for graph algebras

We have already mentioned the utility of K-theory in the study of C^*-algebras. In the case of graph algebras, K-theory provides a powerful tool because it is fairly easy to compute. Cuntz originally calculated the K-theory of Cuntz-Krieger algebras in [14, Proposition 3.1]. This was extended to C^*-algebras of locally finite graphs in [55] and [65], and to row-finite graphs in [78, Theorem 3.2]. The K-theory of infinite graphs with finitely many vertices was calculated in [92, Proposition 2], and the K-theory of arbitrary graphs was computed in [23, Theorem 3.1] by reducing to the row-finite case and applying [78, Theorem 3.2]. K-theory for arbitrary graph algebras was again calculated in [3, Theorem 6.1] where an explicit description of the isomorphism for each K-group is described.

Theorem A.11.5 (K-theory for graph algebras). *Let G be a graph. Also let J be the set of singular vertices of G and let $I := G^0 \backslash J$. Then with respect to the decomposition $G^0 = I \cup J$ the vertex matrix of G will have the form*

$$A_G = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. Then

$$K_0(C^*(G)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \qquad \text{and} \qquad K_1(C^*(G)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$$

where $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}.$

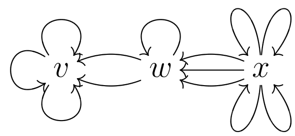
Remark A.11.6. Let A be a finite $n \times n$ matrix with integer entries. Finding the kernel and cokernel of A is quite easy. One simply performs elementary row and column operation to A to obtain a diagonal matrix (remembering that since our matrix is viewed as a mapping on $\mathbb{Z}$ modules, one is only allowed to add *integer* multiples of rows (respectively, columns) to rows (respectively, columns)). This diagonal matrix will have the form

$$\begin{pmatrix} d_1 & & & & & \\ & \ddots & & & & \\ & & d_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

for positive integers $d_1, \ldots d_k$. Now since elementary row and column operations correspond to precomposing and postcomposing by automorphisms, this diagonal matrix will have the same kernel and cokernel as A. Hence

$$\operatorname{coker} A \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k} \qquad \text{and} \qquad \ker A \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k}.$$

Example A.11.7. Let G be the graph



Then the vertex matrix of G is $A_G = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}$ and $A_G^t - I = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. One can perform row and column operations to $A_G^t - I$ to obtain $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and therefore

$$K_0(C^*(G)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad K_1(C^*(G)) \cong \mathbb{Z}.$$

Note that if G is a finite graph, then all of the K-theory information of $C^*(G)$ is contained in the K_0-group. More formally, we have the following:

Corollary A.11.8. *If G is a finite graph, then the following statements hold:*

1. *the K-groups of $C^*(G)$ are finitely generated*
2. *$K_1(C^*(G))$ is a free group*
3. *$K_0(C^*(G)) \cong T \oplus K_1(C^*(G))$ for some finite torsion group T*

Consequently, if G_1 and G_2 are finite graphs, then $K_0(C^*(G_1)) \cong K_0(C^*(G_2))$ implies that $K_1(C^*(G)) \cong K_1(C^*(G))$.

Proof. This follows immediately from the Remark A.11.6. $\square$

Corollary A.11.9. *If G is a graph that has a finite number of vertices (but possibly an infinite number of edges), then* $\operatorname{rank} K_0(C^*(G)) \geq \operatorname{rank} K_1(C^*(G))$.

Proof. If J denotes the singular vertices of G and $I := G^0 \backslash J$, then Theorem A.11.5 gives the following short exact sequence:

$$0 \longrightarrow K_1(C^*(G)) \longrightarrow \mathbb{Z}^I \longrightarrow \mathbb{Z}^I \oplus \mathbb{Z}^J \longrightarrow K_0(C^*(G)) \longrightarrow 0$$

and since I and J are finite we must have $\operatorname{rank} K_0(C^*(G)) \geq \operatorname{rank} K_1(C^*(G))$. □

Corollary A.11.10. *If G is a graph, then $K_1(C^*(G))$ is a free abelian group.*

The above corollary is simply due to the fact that $K_1(C^*(G))$ is isomorphic to the kernel of a map whose domain is $\bigoplus_I \mathbb{Z}$.
Remarkably, this is one of the few conditions that is required of the K-groups of a graph algebra. In fact, Szymański has proven the following in [93, Theorem 3].

Theorem A.11.11. *If (K_0, K_1) is a pair of countable abelian groups with K_1 free, then there exists a row-finite transitive graph G with an infinite number of vertices and with $K_i(C^*(G)) \cong K_i$ for $i = 0, 1$. Furthermore, if $\Xi \in K_0$ we may choose G in such a way that there exists $v \in G^0$ with $[p_v] = \Xi$ in $K_0(C^*(G))$.*

Remark A.11.12. Recall that a finitely generated abelian group is free if and only if it is torsion free. In general, a free abelian group will always be torsion free, however, the converse does not always hold. The abelian group $\mathbb{Q}$, for example, is torsion free but not free.

A.11.3 The classification program and graph algebras

In [83] Rørdam showed that one could classify the simple Cuntz-Krieger algebras up to stable isomorphism by their K_0-group. This classification may be viewed as a special case of the Kirchberg-Phillips Classification Theorem. As we shall see, the

Kirchberg-Phillips Classification Theorem together with Elliott's Theorem gives a complete classification of simple graph algebras.

If G is a graph then Theorem A.8.9 gives conditions for the graph algebra to be simple. If $C^*(G)$ is simple, then the dichotomy of simple graph algebras in Proposition A.8.12 implies that $C^*(G)$ will be AF (if G contains no loops) or purely infinite (if G contains loops). If $C^*(G)$ is AF, then one may use Elliott's Theorem to classify $C^*(G)$; and if $C^*(G)$ is purely infinite, then one may use the Kirchberg-Phillips classification theorem to classify $C^*(G)$. Thus all simple graph algebras are classified by their K-theory. Since we saw in the previous section that the K-theory of a graph algebra is fairly easy to compute, this is an important and useful result.

Remark A.11.13. Let us briefly discuss why purely infinite simple graph algebras satisfy the conditions of the Kirchberg-Phillips Classification Theorem. To begin, since our graphs are countable, we see that graph algebras are separable. In addition, it is shown in [52, Proposition 2.6] that for any directed graph G the crossed product $C^*(G) \times_\alpha \mathbb{T}$ is an AF-algebra. (The proof in [52] is for row-finite graphs, but it should hold for arbitrary graphs as well.) Therefore from the Takesaki-Takai duality theorem (see [71, Theorem 7.9.3]) one has

$$C^*(G) \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (C^*(G) \times_\alpha \mathbb{T}) \times_{\hat{\alpha}} \mathbb{Z}$$

and hence $C^*(G)$ is stably isomorphic to the crossed product of an AF-algebra by $\mathbb{Z}$. It then follows from [9, Corollary 3.2] and [10, Proposition 6.8] that $C^*(G)$ is nuclear, and it follows from [88, Theorem 1.17] and [5, Chapter 23] that $C^*(G)$ satisfies the UCT. Hence the Kirchberg-Phillips Classification Theorem applies to any purely infinite simple graph algebra.

Proposition A.11.14. *If A is any nonunital Kirchberg algebra with $K_1(A)$ free, then* *there exists a row-finite transitive graph for which $A \cong C^*(G)$.*

Proof. It follows from Theorem A.11.11 that there exits a transitive graph G with an infinite number of vertices for which $C^*(G)$ has the same K-theory as A. Now since G is a transitive graph with an infinite number of vertices, it follows from Theorem A.8.9 that $C^*(G)$ is simple and from Proposition A.4.1 that $C^*(G)$ is nonunital. Also, since G must contain a loop, it follows from Proposition A.8.12 that $C^*(G)$ is purely infinite. Thus $C^*(G)$ is a nonunital Kirchberg algebra and the Kirchberg-Phillips Classification Theorem implies that $A \cong C^*(G)$. □

We conclude this section by stating how the K-theory of Exel-Laca algebras is computed, and noticing how similar it is to the computation for graph algebras. The K-theory of Exel-Laca algebras was computed in [28, Theorem 4.5]. Note that for an infinite matrix A indexed by I, it makes sense to multiply elements of $\bigoplus_I \mathbb{Z}$ by A^t but that the vector obtained may not be in $\bigoplus_I \mathbb{Z}$ if A is not a row-finite matrix. Therefore, we will find it necessary to create a larger group $\mathcal{R}_A$ containing $\bigoplus_I \mathbb{Z}$ for our target space.

Theorem A.11.15. *Suppose A is an $I \times I$ $\{0,1\}$-matrix. Let ρ_i denote the i^{th} row of A and let $\mathcal{R}_A$ denote the subring of $\ell^\infty(I)$ generated by the rows ρ_i and the point masses δ_i. Then $A^t - I : \bigoplus_I \mathbb{Z} \to \mathcal{R}_A$ by left-multiplication and*

$$K_0(\mathcal{O}_A) \cong \operatorname{coker}(A^t - I) \qquad \textit{and} \qquad K_1(\mathcal{O}_A) \cong \ker(A^t - I).$$

A.12 Ext for graph algebras

In Chapter 4 of this thesis Ext was computed for C^*-algebras of row-finite graphs. The computation of Ext has also been extended to graphs that are not row-finite.

The following is proven in [23, Theorem 3.1]

Theorem A.12.1. *Let G be a graph that satisfies Condition (L). Also let J be the set of singular vertices of G and let $I := G^0 \backslash J$. With respect to the decomposition $G^0 = I \cup J$ the vertex matrix of G will have the form*

$$A_G = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. Then

$$\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(B - I \ C)$$

where $(B - I \ C) : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \to \prod_I \mathbb{Z}$ by left-multiplication.

A.13 Stable rank one and real rank zero

In the classification program for C^*-algebras two important properties that have been studied are real rank zero and stable rank one. Rieffel was inspired by Bass' stable rank in ring theory to develop a (topological) stable rank for C^*-algebras [82]. Building off of this work, Brown and Pedersen introduced the concept of real rank for C^*-algebras in [8]. In most cases one is primarily concerned with C^*-algebras of stable rank one and with C^*-algebras of real rank zero. The notions of stable rank one and real rank zero for graph algebras have been considered in [46], and the real rank zero results were studied further in [45] and [44].

A.13.1 Stable rank one for graph algebras

Definition A.13.1. A unital C^*-algebra A is said to be of *stable rank one*, written $\mathrm{sr}(A) = 1$, if the set of invertible elements in A is dense in A. A nonunital C^*-algebra has stable rank one if its unitization has stable rank one.

Remark A.13.2. Every unital C^*-algebra of stable rank one is stably finite; that is, its stabilization contains no infinite projections. In addition, if A and B are Morita equivalent C^*-algebras, then $\mathrm{sr}(A) = 1$ if and only if $\mathrm{sr}(B) = 1$.

The following characterization of stable rank one for graph algebras appears in [46, Theorem 3.3].

Theorem A.13.3. *Let G be a row-finite graph. Then* $\mathrm{sr}(C^*(G)) = 1$ *if and only if no loop in G has an exit.*

This theorem implies the following dichotomy for C^*-algebras of cofinal graphs, which appears in [46, Proposition 3.6].

Proposition A.13.4. *Let G be a row-finite graph that is cofinal. Then*

1. $\mathrm{sr}(C^*(G)) = 1$ *if no loops in G have exits*
2. $C^*(G)$ *is purely infinite and simple if there is a loop in G with an exit.*

Proof. If no loop in G has an exit, then $\mathrm{sr}(C^*(G)) = 1$ by Theorem A.13.3. On the other hand, suppose that α is a loop in G with an exit. Then if β is any other loop, we see from cofinality that $s(\beta)$ must connect to the infinite path $\alpha\alpha\alpha\ldots$, and hence β must have an exit. But then G satisfies Condition (L), and by Theorem A.8.2 we have that $C^*(G)$ is simple. Furthermore, since G has a loop Proposition A.8.12 implies that $C^*(G)$ is purely infinite. $\square$

Remark A.13.5. Note that if a cofinal graph contains a loop without an exit, then it must contain a single simple loop.

Recall that if p and q are projections, then we write $p \sim q$ if there exists an element v such that $p = vv^*$ and $q = v^*v$.

Definition A.13.6. A C^*-algebra is said to have the *cancellation property* if for every pair of projections in p and q in $A \otimes \mathcal{K}$ we have

$$p \sim q \qquad \text{if and only if} \qquad [p]_0 = [q]_0 \quad \text{in } K_0(A).$$

Equivalently, A has the cancellation property if for all projections p, q, and r in $A \otimes \mathcal{K}$ one has $p \oplus r \sim q \oplus r$ implies $p \sim q$.

Remark A.13.7. All C^*-algebras of stable rank one have the cancellation property [5, 82].

A.13.2 Real rank zero for graph algebras

Brown and Pedersen introduced the concept of the real rank of a C^*-algebra, an invariant that is an analogue of the topological dimension of a compact Hausdorff space. In practice, one is primarily concerned with the case where the real rank is equal to zero. Brown and Pedersen proved that the class of C^*-algebras of real rank zero includes a surprisingly large number of commonly studied C^*-algebras, including the AF-algebras, von Neumann algebras, the Bunce-Deddens algebras, and the Cuntz algebras $\mathcal{O}_n$.

Definition A.13.8. A unital C^*-algebra A is said to be have *real rank zero*, written $\mathrm{RR}(A) = 0$, if the set of invertible self-adjoint elements of A is dense in the set of self-adjoint elements of A; that is, A_{sa}^{-1} is dense in A_{sa}. A nonunital C^*-algebra is said to have real rank zero if its unitization has real rank zero.

Remark A.13.9. It is a fact that $\mathrm{RR}(A) = 0$ if and only if $\mathrm{RR}(A \otimes \mathcal{K}) = 0$. In particular, if A and B are separable C^*-algebras that are Morita equivalent, then $\mathrm{RR}(A) = 0$ if and only if $\mathrm{RR}(B) = 0$. In addition, if A is a C^*-algebra and I is an ideal in A, then $\mathrm{RR}(A) = 0$ implies that $\mathrm{RR}(I) = 0$ and $\mathrm{RR}(A/I) = 0$.

In [8] Brown and Pedersen proved that the following characterizations of real rank zero hold.

Theorem A.13.10. *If A is a C^*-algebra, then the following conditions are equivalent.*

1. *A has real rank zero*
2. *the self-adjoint elements of A with finite spectrum are dense in the set of all self-adjoint elements of A*
3. *every hereditary C^*-subalgebra of A has an approximate unit (not necessarily increasing) consisting of projections.*

Remark A.13.11. In particular, note that Condition 2 implies that if A is a C^*-algebra of real rank zero, then the linear span of the projections in A is dense in A.

The following result is contained in [46, Theorem 4.3] and [46, Theorem 4.6]. Although the statements of both of these theorems are for locally finite graphs, the proofs work for row-finite graphs as well.

Theorem A.13.12. *Let G be a row-finite graph with no sinks.*

1. *If $\mathrm{RR}(C^*(G)) = 0$, then G satisfies Condition (K).*
2. *If $C^*(G)$ has only finitely many ideals and G satisfies Condition (K), then $\mathrm{RR}(C^*(G)) = 0$.*

Corollary A.13.13. *If G is a finite graph with no sinks, then $\mathrm{RR}(C^*(G)) = 0$ if and only if G satisfies Condition (K).*

This corollary follows from the fact that if G satisfies Condition (K), then the ideals of $C^*(G)$ correspond to saturated hereditary subsets of G^0. Hence if G is finite and satisfies Condition (K), then $C^*(G)$ must have a finite number of ideals.

The result of Theorem A.13.12 was strengthened and extended to C^*-algebras of arbitrary graphs in [43, Theorem 3.7] where the following theorem was proven.

Theorem A.13.14. *Let G be a graph. Then $C^*(G)$ has real rank zero if and only if G satisfies Condition (K).*

We mention that this result was obtained independently by Hong and Szymański in [38].

A.14 Group actions and crossed products

A study of group actions on directed graphs was made in [52]. This work was extended to actions by certain semigroups in [67], and was used to prove a version of the symmetric imprimitivity theorem for graph algebras in [66]. In this section we shall let $E := (E^0, E^1, r, s)$ denote a graph and reserve the symbol G for a group.

Definition A.14.1. Let E and F be graphs. A *graph morphism* $f : E \to F$ is a pair of maps $f = (f^0, f^1)$ where $f^i : E^i \to F^i$ for $i = 0, 1$ are such that $f^0(r(e)) = r(f^1(e))$ and $f^0(s(e)) = r(f^1(e))$ for all $e \in E^1$. If G is a countable group, then G *acts on* E if there is a group homomorphism $\lambda : G \to \mathrm{Aut}(E)$. The action λ is said to *act freely* if it acts freely on vertices; that is, if $\lambda_g(v) = v$ for every $v \in E^0$ implies that $g = 1_G$. Note that in this case G also acts freely on the edges of E.

If G is a countable group which acts freely on E, then by the universal property of E there will exist an induced action of G on $C^*(E)$. Furthermore, if a countable group G acts freely on a graph E, then the quotient E/G has the structure of a directed graph (simply let $E/G = ((E^0/G), (E^1/G), r, s)$ consist of the equivalence classes of vertices and edges under the action of G, together with the range and source maps $r([e]) = [r(e)]$ and $s([e]) = [s(e)]$ which will be well-defined).

In addition, one can label the of edges of a graph E by elements of a countable group G. This amounts to simply defining a function $c : E^1 \to G$.

Definition A.14.2. Let E be a graph, G be a countable group, and $c : E^1 \to G$. The *skew-product graph* is defined to be the graph $E(c) := (G \times E^0, G \times E^1, r, s)$ where

$$r(g, e) = (gc(e), r(e)) \qquad \text{and} \qquad s(g, e) = (g, s(e)).$$

We refer the reader to [52, §2.2] and [68] for examples.

The constructions of quotient graphs and skew-product graphs are linked in the following way: If $c : E^1 \to G$ is a function, then G acts freely on $E(c)$ with $E(c)/G \cong E$. Conversely, if G acts freely on E, then there is a function $c : (E/G)^1 \to G$ such that $(E/G)(c) \cong E$ and this isomorphism is G-equivariant. Because of this, it has been suggested that E may be regarded as the graph theoretical analogue of a principal G-bundle over E/G, and c may be regarded as the analogue of a G-valued cocycle that provides patching data.

The following result is from [52, Theorem 1.1].

Theorem A.14.3 (Kumjian-Pask). *Let E be a locally finite directed graph and suppose that $\lambda : G \to \mathrm{Aut}(E)$ is a free action of a countable group G on the vertices* *of E. Then*

$$C^*(E) \times_\lambda G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G)),$$

where λ also denotes the induced action of G on $C^*(E)$; moreover, if G is abelian then there is an action α of $\hat{G}$ on $C^*(E/G)$ such that

$$C^*(E) \cong C^*(E/G) \times_\alpha \hat{G},$$

and under this isomorphism λ is identified with $\hat{\alpha}$.

An analogue of the Kumjian-Pask Theorem was proven in [67] for free actions of Ore semigroups. For an overview of Ore semigroups in this context one should refer to [68].

Furthermore, the Kumjian-Pask Theorem bears a striking resemblance to a famous theorem of Green which says that the crossed product $C_0(X) \times G$ associated to a free and proper action of G on a locally compact space X is Morita equivalent to $C_0(X/G)$ [33]. In [66] this analogy is pushed further by showing that there are analogues for free actions on graphs of other Morita equivalences associated to free and proper actions on spaces. In particular, the following analogues of the symmetric imprimitivity theorem of [81] and [76] are proven.

Theorem A.14.4. *Suppose a group G acts freely on a directed graph E, and let α be the induced action on $C^*(E)$. Then the reduced crossed product $C^*(E) \times_{\alpha,r} G$ is Morita equivalent to $C^*(G\backslash E)$.*

Theorem A.14.5. *Suppose that we have commuting free actions of two groups G and H on the left and right side of a directed graph E, and let $\alpha : G \to C^*(E/H)$ and $\beta : H \to C^*(G\backslash E)$ denote the induced actions on the C^*-algebras of the quotient graphs. Then $C^*(G\backslash E) \times_\beta H$ is Morita equivalent to $C^*(E/H) \times_\alpha G$.*

In addition, a version of Theorem A.14.4 is proven for reduced crossed products in [66].

A.15 Row-finite graphs and arbitrary graphs

Fowler, Laca, and Raeburn extended the definition of a graph algebra from row-finite graphs to arbitrary graphs in [29]. It is natural to wonder how much larger the class of graph algebras becomes when one includes C^*-algebras of graphs that are not row-finite. It turns out that there are many C^*-algebras of non-row-finite graphs that are not isomorphic to the C^*-algebra of any row-finite graph. However, every graph algebra is Morita equivalent to the C^*-algebra of some row-finite graph.

Proposition A.15.1. *Let G be the graph $\bullet \overset{\infty}{\Longrightarrow} \bullet$ with two vertices and a countably infinite number of edges between them. Then $C^*(G)$ is not isomorphic to the C^*-algebra of any row-finite graph.*

Proof. Suppose that F is a row-finite graph and $C^*(G) \cong C^*(F)$. Since G has finitely many vertices, it follows that $C^*(G)$, and hence $C^*(F)$, is unital. But then F must also have finitely many vertices. Since F is row-finite, this implies that F also has a finite number of edges. Therefore, F is a finite graph. In addition, since G has no loops, it follows that $C^*(G)$, and hence $C^*(F)$, must be AF. But then F must also have no loops. Since F is a finite graph with no loops, it follows from Proposition A.4.3 that $C^*(F) \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ for some $n_1, \ldots, n_k$. But $C^*(G)$ contains infinitely many nonzero partial isometries $\{s_e : e \in G^1\}$ with mutually orthogonal ranges, contradicting the fact that it is a finite-dimensional C^*-algebra. $\square$

Remark A.15.2. The above argument can be used to show that whenever G is a graph with a finite number of vertices, an infinite number of edges, and no loops, then $C^*(G)$

is not isomorphic to the C^*-algebra of a row-finite graph. In particular, this shows that there are many UHF algebras that are in the class of C^*-algebras of arbitrary graphs, but not in the class of C^*-algebras of row-finite graphs.

In the attempt to extend results obtained for C^*-algebras of row-finite graph to arbitrary graph algebras, many different techniques have been employed. One approach is found in [78] where certain subalgebras of a graph algebra are realized as C^*-algebras of finite graphs. This allows one to view a graph algebra as an increasing union (or more generally, a direct limit) of C^*-algebras of finite graphs. Another approach is taken in [22] where an operation called *desingularization* was described. This operation allows one to turn an arbitrary graph into a row-finite graph, while at the same time preserving the Morita equivalence class of the associated C^*-algebra. We shall describe these two methods in the following sections.

A.15.1 Subalgebras of graph algebras

Definition A.15.3. Let G be a graph with no sinks and let $F \subseteq G^1$ be a finite set. We define G_F to be the finite graph given by

$$G_F^0 := F \cup (r(F) \cap s(F) \cap s(G^1 \backslash F)), \qquad s(e,f) = e,$$
$$G_F^1 := \{(e,f) \in F \times G_F^0 : r(e) = s(f)\}, \qquad r(e,f) = f.$$

The following result was proven in [78, Lemma 1.2]

Theorem A.15.4. *Let G be a graph and let $F \subseteq G^1$ be a finite set of edges. Then $C^*(G_F)$ is naturally isomorphic to the C^*-subalgebra of $C^*(G)$ generated by $\{s_e : e \in F\}$.*

This theorem implies that if G is a graph with no sinks or sources and we write

$G^1 = \bigcup_{n=1}^{\infty} F_n$ as the increasing union of finite subsets F_n, then $C^*(G) = \bigcup_{n=1}^{\infty} C^*(G_F)$ will be the increasing union of C^*-algebras of finite graphs. (It is important to note that the graph G_F may have sinks even if the graph G does not.) This result allows one to apply many results for row-finite graphs to the union and thereby extend them to arbitrary graphs. In particular, this method was used to prove a version of the Cuntz-Krieger Uniqueness Theorem for C^*-algebras of arbitrary graphs [78, Theorem 1.5].

Remark A.15.5. A similar method is described in [78, §2] for realizing certain C^*-subalgebras of Exel-Laca algebras as C^*-algebras of finite graphs. These approximation techniques can then be used to extend results from C^*-algebras of finite graphs to Exel-Laca algebras. This technique was used to prove a version of the Gauge-Invariant Uniqueness Theorem for Exel-Laca algebras [78, Theorem 2.7] and to compute the K-theory of Exel-Laca algebras [78, §4].

A.15.2 Desingularization

Another method for studying C^*-algebras of arbitrary graphs is to use an operation called desingularization that was introduced in [22]. Desingularization transforms an arbitrary graph into a row-finite graph with no sinks, while at the same time preserving Morita equivalence of the associated C^*-algebra as well as the loop structure and path space of the graph. Consequently, it is a powerful tool in the analysis of graph algebras because it allows one to apply much of the machinery that has been developed for row-finite graph algebras to arbitrary graph algebras.

Desingularization was motivated by the process of "adding a tail to a sink" that is described in [4]. In fact, this process is actually a special case of desingularization. The difference is that now we not only add tails at sinks, but we also add (more complicated) tails at vertices that emit infinitely many edges. Consequently, we shall see that vertices that emit infinitely many edges will often behave very similarly to sinks in the way that they affect the associated C^*-algebra. In fact, in many theorems one can take the result for row-finite graphs and replace the word "sink" by the phrase "sink or vertex that emits infinitely many edges" to get the corresponding result for arbitrary graphs.

Given a graph G we shall construct a graph F, called a desingularization of G, with the property that F has no singular vertices and $C^*(G)$ is isomorphic to a full corner of $C^*(F)$. Loosely speaking, we will build F from G by replacing every singular vertex v_0 with its own infinite path, and then redistributing the edges of $s^{-1}(v_0)$ along the vertices of the infinite path. Note that if v_0 happens to be a sink, then $|s^{-1}(v_0)| = 0$ and there are no edges to redistribute. In that case our procedure will coincide with the process of adding an infinite tail to a sink described in [4, (1.2)].

Definition A.15.6. Let G be a graph with a singular vertex v_0. We *add a tail* to v_0 by performing the following procedure. If v_0 is a sink, we add a graph of the form

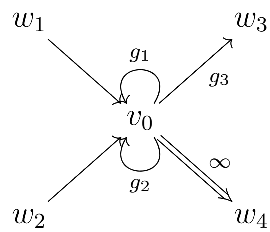
$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots \tag{A.3}$$

as described in [4, (1.2)]. If v_0 is an infinite emitter we first list the edges $g_1, g_2, g_3, \ldots$ of $s^{-1}(v_0)$. Then we add a graph of the form shown in (A.3), remove the edges in $s^{-1}(v_0)$, and for every $g_j \in s^{-1}(v_0)$ we draw an edge f_j from v_{j-1} to $r(g_j)$.

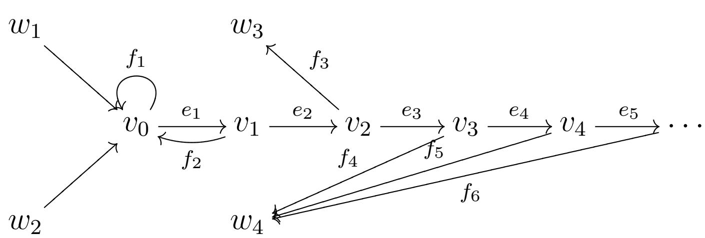
Note that different orderings of the edges of $s^{-1}(v_0)$ may give rise to nonisomorphic graphs via the above procedure.

Definition A.15.7. If G is a directed graph, a *desingularization of* G is a graph F obtained by adding a tail at every singular vertex of G.

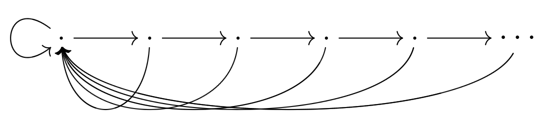
Example A.15.8. Suppose we have a graph G containing this fragment:



where the double arrow labeled ∞ denotes a countably infinite number of edges from v_0 to w_4. Let us label the edges from v_0 to w_4 as $\{g_4, g_5, g_6, \ldots\}$. Then a desingularization of G is given by the following graph F:



Example A.15.9. If G is the $\mathcal{O}_\infty$ graph (one vertex with infinitely many loops), a desingularization F looks like this:



Example A.15.10. The following graph was mentioned in [29, Remark 11]:

$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow v_0 \overset{\infty}{\Longrightarrow} \cdot \longrightarrow \cdot \longrightarrow \cdots$$

A desingularization of it is:

$$\begin{tikzcd}
\cdots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & v_0 \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdot \arrow[r] & \cdots \\
 & & & v_1 \arrow[d] \arrow[ur] & & & \\
 & & & v_2 \arrow[d] \arrow[uur] & & & \\
 & & & v_3 \arrow[d] \arrow[uuur] & & & \\
 & & & \vdots \arrow[uuuur] & & &
\end{tikzcd}$$

It is crucial that desingularizing a graph preserves connectivity, path space, and loop structure in the appropriate senses, and this will turn out to be the case. This is made explicit in [22, Lemma 2.6], [22, Lemma 2.7], and [22, Lemma 2.8]. In addition, it turns out that $C^*(F)$ is Morita equivalent to $C^*(G)$. The following was proven in [22, Theorem 2.11].

Theorem A.15.11. *Let G be a graph and let F be a desingularization of G. Then $C^*(G)$ is isomorphic to a full corner of $C^*(F)$. Consequently, $C^*(G)$ and $C^*(F)$ are Morita equivalent.*

Desingularization allows one to extend many results for row-finite graphs to arbitrary graphs in a fairly straightforward manner. In particular it was used in [22] to obtain generalizations of the Cuntz-Krieger Uniqueness Theorem [22, Corollary 2.12] as well as characterizations of when a graph algebra will be AF [22, Corollary 2.13], purely infinite [22, Corollary 2.14], and simple [22, Corollary 2.15]. In addition, desingularization can be used in more sophisticated ways to analyze the ideal structure of a graph algebra and to obtain results which are nontrivial generalizations of the row-finite results. This was done for graphs satisfying Condition (K) in [22, §3 and §4] to obtain descriptions of the ideal structure [22, Theorem 3.5] and the primitive ideal space [22, Theorem 4.10].

Remark A.15.12. Many of the results mentioned in the previous paragraph have also been obtained using direct methods rather than desingularization (see [3], for example).

A.16 Exel-Laca algebras and their relationship with graph algebras

Since Exel-Laca algebras are generalizations of the Cuntz-Krieger algebras, they are closely related to graph algebras. However, the classes of Exel-Laca algebras and graph algebras are incomparable; that is, there exist C^*-algebras in each class that are not in the other. Consequently, the relationship between the two is fairly subtle.

To begin, let us first show that the the classes are distinct. The following argument comes from [78, Remark 4.4].

Proposition A.16.1 (An Exel-Laca algebra that is not a graph algebra). *If*

$$A = \begin{pmatrix} \begin{smallmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{smallmatrix} & \cdots \\ \vdots & \ddots \end{pmatrix}$$

then the Exel-Laca algebra $\mathcal{O}_A$ is not isomorphic to a graph algebra.

Proof. We see that $\mathcal{O}_A$ is unital: indeed, $s_1^* s_1 + s_2 s_2^* = 1$. Furthermore, from the K-theory computations of [78, Theorem 4.1] and [78, Example 4.2] we see that $K_0(\mathcal{O}_A) \cong 0$ and $K_1(\mathcal{O}_A) \cong \mathbb{Z}$. But from Corollary A.11.9 we see that any unital graph algebra

$C^*(G)$ has rank $K_0(C^*(G)) \geq$ rank $K_1(C^*(G))$. Thus $\mathcal{O}_A$ is not isomorphic to any graph algebra. $\square$

The ideas in the following argument were suggested by Wojciech Szymański.

Proposition A.16.2. *Let G be the graph*

$$v_1 \overset{\infty}{\Longrightarrow} v_2 \circlearrowleft$$

Then $C^*(G)$ is not isomorphic to any Exel-Laca algebra.

Proof. Recall that a character for $C^*(G)$ is a nonzero homomorphism $\epsilon : C^*(G) \to \mathbb{C}$. We shall show that there is a unique character on $C^*(G)$. Let $\{s_e, p_v\}$ be a generating Cuntz-Krieger G-family. Note that $\{v_2\}$ is a saturated hereditary set and that $C^*(G)/I_{v_2} \cong \mathbb{C}$. Thus the projection $\pi : C^*(G) \to C^*(G)/I_{v_2}$ is a character. We shall now show that this character is unique. Let $\epsilon : C^*(G) \to \mathbb{C}$ be a character and set $I := \ker \epsilon$. Then I is a nonzero ideal and $H := \{v \in G^0 : p_v \in I\}$ is a saturated hereditary set of vertices. By the Cuntz-Krieger Uniqueness Theorem $\ker \epsilon$ contains one of the p_v's, and thus H is nonempty. Now the only nonempty saturated hereditary subsets of G are $\{v_1, v_2\}$ and $\{v_2\}$. Since ϵ is nonzero, we cannot have $H = \{v_1, v_2\}$. Thus we must have $H = \{v_2\}$. Now since $C^*(G)$ is generated by $\{s_e : e \in G^1\} \cup \{p_{v_1}\} \cup \{p_{v_2}\}$, and

$$\epsilon(p_{v_2}) = 0 \qquad \text{and} \qquad \epsilon(s_e) = \epsilon(s_e s_e^* s_e) = \epsilon(s_e)\epsilon(p_{v_2}) = 0 \quad \text{for all } e \in G^1$$

we see that ϵ is completely determined by its value on p_{v_1}. Because p_{v_1} is a projection, $\epsilon(p_{v_1}) = 1$. Thus ϵ is unique.

Now if $C^*(G)$ was an Exel-Laca algebra, then $C^*(G)$ would be generated by an

Exel-Laca family $\{S_i\}$. Let γ be the gauge action on this Exel-Laca algebra. Because there is a unique character ϵ on $C^*(G)$, we see that $\epsilon \circ \gamma_z = \epsilon$ for all $z \in \mathbb{T}$. Also, since ϵ is nonzero, $\epsilon(S_i) \neq 0$ for some i. But then $\epsilon(S_i) = \epsilon(\gamma_z(S_i)) = z\epsilon(S_i)$ for all $z \in \mathbb{T}$ which is a contradiction. $\square$

Note that in the above example, it is the fact that the infinite emitter is a source that prevents the graph algebra from being an Exel-Laca algebra. A similar problem occurs when sinks are present in a graph. However, if a graph has no sinks and no sources, then the C^*-algebra associated to it will be an Exel-Laca algebra in a natural way. The following result is proven in [29, Theorem 10].

Theorem A.16.3. *Let G be a graph with no sinks or sources and let B_G be the edge matrix of G. Then $C^*(G)$ is canonically isomorphic to $\mathcal{O}_{B_G}$.*

Remark A.16.4. If $\mathcal{O}_A$ is an Exel-Laca algebra and A is the edge matrix of a graph G, then the above result shows that $\mathcal{O}_A$ will be isomorphic to the graph algebra $C^*(G)$. Unfortunately, not all $\{0,1\}$-matrices arise as the edge matrix of a graph. For example, the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is not the edge matrix of any graph. Thus this technique can only be used to show that very special types of Exel-Laca algebras are graph algebras.

In general, if $\mathcal{O}_A$ is an Exel-Laca algebra, then one may form the graph $\mathrm{Gr}(A)$ whose vertices are the index set I of the matrix A and with one edge from i to j if and only if $A(i,j) = 1$. If there exists an edge from i to j we shall denote it by (i,j).

Theorem A.16.5. *Let A be a $\{0,1\}$-matrix with no zero rows. Then $S_{(i,j)} := S_iS_jS_j^*$ and $P_i := S_iS_i^*$ define a Cuntz-Krieger $\mathrm{Gr}(A)$-family, and there is an isomorphism $\phi : C^*(\mathrm{Gr}(A)) \to C^*(\{S_{(i,j)}, P_i\})$ with $\phi(s_{(i,j)}) = S_{(i,j)}$ and $\phi(p_i) = P_i$.*

Furthermore, if A is row-finite, then $C^*(\{S_{(i,j)}, P_i\}) = \mathcal{O}_A$ and $C^*(\mathrm{Gr}(A)) \cong \mathcal{O}_A$.

Proof. It is straightforward to verify that $\{S_{(i,j)}, P_i\}$ is a Cuntz-Krieger $\mathrm{Gr}(A)$-family. Therefore by the universal property of $C^*(\mathrm{Gr}(A))$ there is a homomorphism $\phi : C^*(\mathrm{Gr}(A)) \to \mathcal{O}_A$ with $\phi(s_{(i,j)}) = S_{(i,j)}$ and $\phi(p_i) = P_i$. If γ denotes the gauge action on $C^*(\mathrm{Gr}(A))$ and β denotes the gauge action on $\mathcal{O}_A$, then by checking on generators one can see that $\phi \circ \gamma = \beta \circ \phi$. Hence by the Gauge-Invariant Uniqueness Theorem ϕ is injective, and is an isomorphism onto $C^*(\{S_{(i,j)}, P_i\}) \subseteq \mathcal{O}_A$.

To see that $C^*(\{S_{(i,j)}, P_i\}) = \mathcal{O}_A$ when A is row-finite, note that for any index j the finite sum $S_j = \sum_{A(i,j)=1} S_{(i,j)}$ will in $C^*(\{S_{(i,j)}, P_i\})$. Hence this subalgebra is all of $\mathcal{O}_A$. $\square$

Remark A.16.6. Note that in the row-finite case the isomorphism $\phi : C^*(\mathrm{Gr}(A)) \to \mathcal{O}_A$ is not canonical. In fact, this isomorphism comes from an isomorphism of $C^*(\mathrm{Gr}(A))$ with the C^*-algebra of its dual graph (see [4, Corollary 2.5]).

Remark A.16.7. When A is not row-finite, the subalgebra $C^*(\{S_{(i,j)}, P_i\}) \subseteq \mathcal{O}_A$ may be quite different from $\mathcal{O}_A$. In [22] an example is produced in which this subalgebra and $\mathcal{O}_A$ have different K-theory, so in particular we see that they need not even be Morita equivalent.

Remark A.16.8. If $\mathcal{O}_A$ is an AF-algebra, then $\mathrm{Gr}(A)$ will not contain any loops. Furthermore, since A contains no zero rows, it follows that $\mathrm{Gr}(A)$ has no sinks, and therefore $\mathrm{Gr}(A)$ must have an infinite number of vertices. But then $C^*(\mathrm{Gr}(A))$ is not a finite-dimensional C^*-algebra, and since $C^*(\mathrm{Gr}(A))$ is isomorphic to a subalgebra of $\mathcal{O}_A$, it follows that $\mathcal{O}_A$ is also not finite-dimensional. Consequently, no Exel-Laca algebras are finite-dimensional and we have exhibited a whole class of graph algebras that are not Exel-Laca algebras: If G is a finite graph with no loops, then $C^*(G)$ is a finite-dimensional C^*-algebra by Theorem A.4.3 and hence cannot be an Exel-Laca algebra.

These relationships between graph algebras and Exel-Laca algebras are summarized in Diagram 1.1.

Remark A.16.9. In [97] and [98] a generalization of a graph, called an *ultragraph*, was defined and it was described how to associate a C^*-algebra to it. The class of ultragraph algebras contains all graph algebras and all Exel-Laca algebras as well as C^*-algebras that are in neither of these classes. We saw in Section A.15.2 that every graph algebra is Morita equivalent to the C^*-algebra of a row-finite graph with no sinks. It is also shown in [97] that any ultragraph algebra is Morita equivalent to an Exel-Laca algebra. It is currently an open question as to whether every Exel-Laca algebra is Morita equivalent to a graph algebra.

A.17 C^*-algebras related to graph algebras

A.17.1 C^*-algebras that arise as graph algebras

One reason that graph algebras are so important is the fact that many interesting C^*-algebras are either isomorphic to graph algebras or closely related to graph algebras. The following is just a partial list of such C^*-algebras.

- The Cuntz algebras, the Cuntz-Krieger algebras, the compact operators $\mathcal{K}$, the Toeplitz algebra $\mathcal{T}$, $M_n(C(\mathbb{T}))$ for any $n \geq 1$, and all finite-dimensional C^*-algebras are isomorphic to graph algebras.

- In [17] and [18] Doplicher and Roberts associated an algebra $\mathcal{O}_\rho$ to a special unitary representation $\rho : K \to SU(n)$ of a compact group K. Cuntz-Krieger algebras arise naturally in the computation of the K-theory of $\mathcal{O}_\rho$ for finite groups K. When K is an infinite compact group Cuntz-Krieger algebras of certain infinite matrices arise, and this was the original motivation for the development of graph algebras. It is shown in [55, §7] that every Doplicher-Roberts algebra $\mathcal{O}_\rho$ is isomorphic to a full corner of a graph algebra.

- Drinen has shown in [21] that every AF-algebra is Morita equivalent to the C^*-algebra of a row-finite graph.

- Hong and Szymański have shown in [36] that the C^*-algebras of continuous functions on quantum spheres, quantum real projective spaces, and quantum complex projective spaces are all isomorphic to graph algebras.

- It follows from [93, Theorem 3] that any nonunital Kirchberg algebra with free K_1-group is isomorphic to the C^*-algebra of a row-finite transitive graph. (This was discussed in §A.11.3 and Proposition A.11.14.) There are many unital Kirchberg algebras that are not isomorphic to graph algebras since any unital graph algebra $C^*(G)$ has $\operatorname{rank} K_0(C^*(G)) \geq \operatorname{rank} K_1(C^*(G))$. However, the stabilization of a unital Kirchberg algebra will be a nonunital Kirchberg algebra, and thus any Kirchberg algebra will free K_1-group will be Morita equivalent to the C^*-algebra of a row-finite transitive graph.

A.17.2 Cuntz-Pimsner algebras

Let A be a C^*-algebra, let X be a right Hilbert A-module, and let $\phi : A \to \mathcal{L}(X)$ be a homomorphism. Then $a \cdot x := \phi(a)x$ defines a left action of A on X, and we call X a *Hilbert bimodule* over A. In [73] Pimsner described a way to construct a C^*-algebra $\mathcal{O}_X$ from a Hilbert bimodule X. These Cuntz-Pimsner algebras comprise an extremely large class of C^*-algebras. Not only do they generalize the Cuntz-Krieger algebras, but they also contain the C^*-algebras of graphs with no sinks, the Exel-Laca algebras, and crossed products by $\mathbb{Z}$.

The algebras $\mathcal{O}_X$ were originally defined in a concrete way: Pimsner first introduced a Toeplitz algebra $\mathcal{T}_X$ and took $\mathcal{O}_X$ as a particular quotient of $\mathcal{T}_X$. In his analysis the ideal $J(X) := \phi^{-1}(\mathcal{K}(X))$ played an important role. Since that time Muhly and Solel [60] have also considered generalizations of Cuntz-Pimsner algebras. Given an ideal $K \triangleleft J(X)$, they define a C^*-algebra $\mathcal{O}(K, X)$ called the *relative Cuntz-Pimsner algebra determined by* K. It turns out that $\mathcal{O}(\{0\}, X) = \mathcal{T}_X$ and $\mathcal{O}(J(X), X) = \mathcal{O}_X$.

The ideals $J(X)$ and K have an interesting interpretation in the context of graph algebras. If $G = (G^0, G^1, r, s)$ is a graph with no sinks, then there exists a Hilbert bimodule $X(G)$ over the C^*-algebra $A := C_0(G^0)$ for which $\mathcal{O}_{X(G)} \cong C^*(G)$ [31, §4]. Furthermore,

$$J(X) = \overline{\text{span}}\{\delta_v : v \in G^0 \text{ and } v \text{ is not an infinite emitter}\},$$

where δ_v denotes the point mass at v. Thus $J(X)$ is spanned by the δ_v's where v ranges over the vertices at which a relation is imposed by Condition 2 in the Cuntz-Krieger relations. More generally, if $K \triangleleft J(X)$, then K will have the form $K = \overline{\text{span}}\{\delta_v : v \in V\}$ for some subset V of vertices that are not infinite emitters. It then turns out that $\mathcal{O}(K, X(G))$ is a "relative graph algebra"; that is, it is generated by projections $\{p_v : v \in G^0\}$ and partial isometries $\{s_e : e \in G^1\}$ satisfying the same conditions as for the graph algebra, except that Condition 2 of the Cuntz-Krieger relations is imposed only at the vertices in V.

In addition, while C^*-algebras of graphs with sinks cannot be obtained as $\mathcal{O}_X$'s, they can be obtained as relative Cuntz-Pimsner algebras: one simply chooses V to be all vertices except for sinks and infinite emitters, so that the relation in Condition 2 is imposed at precisely the vertices which emit a finite and nonzero number of edges.

It has been shown in [30] that the ideals of a relative Cuntz-Pimsner algebra $\mathcal{O}(K, X)$ may be studied by methods analogous to those in [4] and that results similar to those for graph algebras often hold. In particular, a gauge-invariant uniqueness theorem for Cuntz-Pimsner algebras $\mathcal{O}_X$ is proven in [30, Theorem 4.1].

A.17.3 Higher rank graph algebras

In [53] Kumjian and Pask defined a combinatorial object called a *higher rank graph* and described a way to associate a C^*-algebra to it. If Λ is a higher rank graph, then $C^*(\Lambda)$ is generated by a family of partial isometries satisfying relations similar to the Cuntz-Krieger relations. These higher rank graph algebras generalize the C^*-algebras of row-finite graphs with no sinks (which arise as rank 1 graphs in the terminology of Kumjian and Pask). It turns out that a rank k graph has a canonical action of $\mathbb{T}^k$ on it, which is analogous to the gauge action of graph algebras. In [53, §3] a version of the gauge-invariant uniqueness theorem is proven for higher rank graph algebras. In addition, a version of the Cuntz-Krieger uniqueness theorem and conditions for simplicity of a higher rank graph algebra are proven in [53, §4], however, Condition (L) must be replaced by an aperiodicity condition.

Higher rank graphs were studied further in [77] where the defining Cuntz-Krieger relations were modified to deal with higher rank graphs with sources. A local convexity condition is also described which characterizes higher rank graphs that admit nontrivial Cuntz-Krieger families. In addition, versions of the uniqueness theorems and classification of ideals for certain higher rank algebras were proven [77, §4 and §5].

"Dissertations are not finished. They are abandoned."

— *Fred Brooks*

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